MARKSCHEME

May 2004

FURTHER MATHEMATICS

Standard Level

Paper 2

12 pages

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Errors made at any step of a solution can affect all working that follows. To limit the severity of the penalty, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent working;

(iii) award ***M*** marks for a correct method, and ***A*****(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
\$ 600 × 1.02	***M1***	Amount earned = \$ 600 × 1.02	√	***M1***
= \$ 612	***A1***	= \$602	×	***A0***
\$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	√	***M1***
= \$ 630.36	***A1***	= \$ 620.06	√	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative solutions, including graphic display calculator alternative solutions are indicated by **OR**. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect ie, once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

5 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

There are two types of accuracy error. Candidates should be penalized **once only IN THE PAPER** for an accuracy error **(AP)**. Award the marks as usual then write -1**(AP)** against the answer and also on the **front** cover.

Rounding errors: only applies to final answers not to intermediate steps.

Level of accuracy: when this is not specified in the question the general rule *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figures* applies.

- If a final correct answer is incorrectly rounded, apply the AP.

  OR

- If the level of accuracy is not specified in the question, apply the AP for answers not given to 3 significant figures. (Please note that this has changed from May 2003).

Incorrect answers are wrong, and the accuracy penalty should not be applied to incorrect answers.

Examples

A question leads to the answer 4.6789….
- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : both should be penalised the first time this type of error occurs.
- 4.67 is incorrectly rounded - penalise on the first occurrence.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Calculator penalties

Candidates are instructed to write the make and model of their calculator on the front cover. Please apply the following penalties where appropriate.

(i) Illegal calculators

If candidates note that they are using an illegal calculator, please report this on a PRF, and deduct 10 % of their overall mark.. Note this on the front cover.

(ii) Calculator box not filled in.

Please apply a calculator penalty (***CP***) of 1 mark if this information is not provided. Note this on the front cover.

1. (i) (a) Since $\sum_{1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0 \Rightarrow u_n < 1$ for large *n*, and since $v_n > 0$, ***(R1)***

then $u_n v_n < v_n$. ***(R1)***

(b) By comparison, and since $\sum_{1}^{\infty} v_n$ converges, then $\sum_{1}^{\infty} u_n v_n$ must converge. ***(R1)***

Similarly, $u_n < 1 \Rightarrow u_n^2 < u_n$ and $\sum_{1}^{\infty} u_n^2$ must converge because $\sum_{1}^{\infty} u_n$ converges. ***(R1)***

[4 marks]

(ii) (a) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots$ ***(A1)***

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots + c$, but $\ln 1 = 0$, hence $c = 0$ which extends ***(M1)(A1)(A1)***

for $-1 < x \le 1$. ***(A1)***

(b) $\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots$, hence ***(A1)***

$\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) =$ ***(M1)***

$\dfrac{1}{2}\left(\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\right) - \left(-x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots\right)\right) =$ ***(M1)***

$\dfrac{1}{2}\left(2x + \dfrac{2x^3}{3} + \dfrac{2x^5}{5} + \ldots\right) = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$ ***(A1)***

(c) $\dfrac{1}{30}\left[1 + \dfrac{4}{1.1} + \dfrac{2}{1.2} + \ldots + \dfrac{4}{1.9} + \dfrac{1}{2}\right] = 0.693150$ ***(M1)(A1)***

(d) $\left| E_s \right| \le \dfrac{M\,(b-a)^5}{180n^4} = \dfrac{24 \times 1}{180 \times 10^4} = 0.000013$ ***(M1)(A1)***

(e) $f(x) = \dfrac{1}{1+x} - x$, to use a fixed-point iteration to find the zero the equation can be re-written as follows:

$f(x) = 0 \Rightarrow \dfrac{1}{1+x} = x$, and hence, the fixed point iteration will be $x_{n+1} = \dfrac{1}{1+x_n}$. ***(M1)(A1)***

Starting with $x_0 = 1$, we receive the following

$x_1 = 0.5, x_2 = 0.666\ldots, x_3 = 0.6, x_4 = 0.625, \ldots, x_{16} = 0.618034$. ***(M1)(A1)***

[17 marks]

Total [21 marks]

2. (i) (a) $pxq = r \Rightarrow p^{-1}pxqq^{-1} = p^{-1}rq^{-1}$ ***(M1)***

$\Rightarrow exe = p^{-1}rq^{-1} \Rightarrow x = p^{-1}rq^{-1}$ ***(A1)***

(b) $px^2 = q \Rightarrow px^2x = qx \Rightarrow px^3 = qx$ ***(M1)***

$\Rightarrow pe = qx \Rightarrow p = qx$ ***(A1)***

$\Rightarrow q^{-1}p = q^{-1}qx \Rightarrow q^{-1}p = ex$ ***(M1)***

$\Rightarrow x = q^{-1}p$ ***(A1)***

[6 marks]

(ii) (a) $(xy)(yx) = x(y^2)x = xex = x^2 = e$ ***(M1)***

Also, $(yx)(xy) = e$ by the same way. ***(M1)***

Now, $(xy)^{-1} = yx$ and $(xy)^2 = e \Leftrightarrow (xy)(xy) = e \Leftrightarrow (xy)^{-1} = xy$

therefore, $xy = yx$ and the group is Abelian. ***(M1)(R1)***

(b) (i) Assume that $H \cap aH \neq \varnothing$, then there is at least one element $x \in H \cap aH$, therefore $x \in H$ and $x \in aH$, ***(M1)***

hence $x = ah$ for some $h \in H$, so $xh = ah^2 = a$, but since $x \in H$ and ***(R1)***

$h \in H$ so is $xh \in H$ and hence $a \in H$, but we are given that $a \notin H$, ***(M1)***

which is a contradiction and therefore $H \cap aH = \varnothing$. ***(R1)***

(ii) For $H \cup aH$ to be a subgroup, it is enough to prove that for every two elements x and y in $H \cup aH$, then xy^{-1} is in $H \cup aH$. ***(R1)***

If $x, y \in H$, it is obvious that $xy^{-1} \in H$ since H is a subgroup of G. ***(R1)***

If $x \in H$ and $y \in aH$, then $xy^{-1} = xy$ since $y^{-1} = y$ as proved above. ***(M1)***

Hence $xy^{-1} = xy = xah = axh = az \in aH \subset H \cup aH$. ***(R1)***

If $x, y \in aH$, then $xy^{-1} = xy = ahak = a^2hk = hk \in H \subset H \cup aH$. ***(R1)***

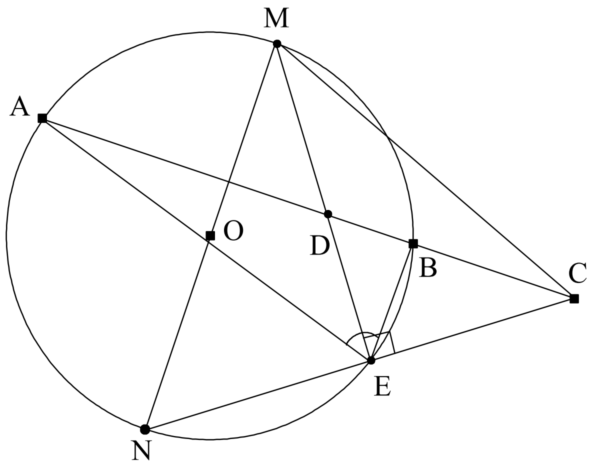
(iii) $n(aH) = n(H)$ since we can set up a 1-1 correspondence between their elements defined as $x \leftrightarrow ax$. ***(M1)***

$n(H \cup aH) = n(H) + n(aH) - n(H \cap aH) = n(H) + n(H) - 0 = 2n(H)$. ***(M1)(AG)***

[15 marks]

Total [21 marks]

3.



(i) (a) Since the diameter is perpendicular to AB, then it is also the perpendicular bisector of that chord. Hence the locus of I is the perpendicular bisector of AB. ***(R2)***

(b) Join the following:
EB and EA. ***(M1)***
$\hat{BEM} =$ half arc BM, $\hat{DEA} =$ half arc MA, ***(R1)***
but the arcs have equal measures, hence the angles are congruent. ***(R1)***
Consequently, ED is the bisector of the vertex angle of triangle BEA. ***(R1)***
Also, EC is perpendicular to ED since $\hat{CED}$ complements $\hat{DEN}$ which is a right angle. So, EC is the external bisector of $\hat{BEA}$. ***(R1)***
By the bisector theorem, ABDC is a harmonic division. And since A, B, and C are fixed, so is D. ***(R1)(AG)***

(c) As shown above, $\hat{DEC}$ is a right angle, and since both C and D are fixed, ***(R1)***
therefore the locus of E is a semicircle with diameter CD. ***(R1)(A1)***

[11 marks]

(ii) (a) The equation of the tangent to the hyperbola at a point (x_0, y_0) is
$y - y_0 = +\frac{b^2}{a^2} \times \frac{x_0}{y_0}(x - x_0)$ since the derivative is $+\frac{b^2}{a^2} \times \frac{x_0}{y_0}$. ***(M1)(A1)***

Let x_M represent the x-intercept of the tangent, hence $x_M = \frac{a^2}{x_0}$. ***(A1)***

The equation of the normal is similarly found: $y - y_0 = -\frac{a^2}{b^2} \times \frac{y_0}{x_0}(x - x_0)$, and ***(M1)***

$x_N = \frac{c^2 x_0}{a^2}$ is the x-intercept of the normal, where c is half the distance between the two foci. ***(A1)***

(b) Therefore $x_M x_N = \frac{a^2}{x_0} \times \frac{c^2 x_0}{a^2} = c^2 = OG \times OF$. ***(M1)(A1)***
Hence, the points form a harmonic division. ***(AG)***

[7 marks]

Total [18 marks]

4. (i) We know that 2 has 9 as an inverse mod 17, *i.e.* $2\times 9\equiv 1(\text{mod}17)$. ***(M1)***

Multiply by 9: $9\times 2x\equiv 9\times 7(\text{mod}17)\Rightarrow x\equiv 12(\text{mod}17)$, hence x = 12, 29, 46. ***(M1)(A1)(A1)***

[4 marks]

(ii) (a) $a_n - b_n = 2a_{n-1} \Rightarrow b_n = a_n - 2a_{n-1}$ ***(M1)***

by substituting into the first equation:

$a_n = 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) = 5a_{n-1} - 4a_{n-2}$ ***(M1)(AG)***

(b) From the first equation we can get another initial condition for a_n.

$a_1 = 3a_0 + 2b_0 = 7$. ***(M1)***

Now we solve for a_n the usual way:

The characteristic equation is $r^2 - 5r + 4 = 0$ with roots 1 and 4. ***(M1)(A1)***

The solution after solving for the arbitrary constants is $a_n = 2\times 4^n - 1$. ***(M2)(A1)***

Finally, since $b_n = a_n - 2a_{n-1} = 1 + 4^n$. ***(M1)(A1)***

[10 marks]

(iii) First put A into S (path), with label 0, and vertex B is labelled 4(A), vertex C is labelled 3(A). Now, D is labelled 6(AC) and E is labelled 7(ACD), then F is labelled 11(ACD) and G labelled 12(ACDE), and lastly H is labelled 16(ACDEG). ***(M1)(M1)*** ***(M1)(A1)***

[4 marks]

Total [18 marks]

5. *Answers may vary slightly depending on whether a table or a calculator is used.*

(i) Let T be the life of a resistor in thousands of hours. $T \sim N(14, 9)$.

(a) $\mathrm{P}(T > 12\,000) = \mathrm{P}\left(Z > \dfrac{12\,000 - 14\,000}{3\,000}\right) = \mathrm{P}\left(Z > -\dfrac{2}{3}\right) = 0.748$ (accept 0.749). ***(M1)(A1)***

(b) (i) $\sum_{1}^{5} T \sim \mathrm{N}(70\,000, 45 \times 10^6)$ ***(A1)***

$$\mathrm{P}\left(\sum T > 60\,000\right) = \mathrm{P}\left(Z > \frac{60\,000 - 70\,000}{3\,000\sqrt{5}}\right) = 0.932$$ ***(M1)(A1)***

(ii) $\mathrm{P}(T < 12\,000) = 1 - 0.748 = 0.252$ ***(M1)(A1)***

Let N be the number of resistors that last less than 12 thousand.

$N \sim \mathrm{B}(5, 0.252)$ ***(A1)***

$\mathrm{P}(N \geq 2) = \sum_{i=2}^{5} \mathrm{P}(N = i) = 1 - \sum_{i=0}^{1} \mathrm{P}(N = i) = 0.372$ ***(M1)(A1)***

(iii) $\bar{T} \sim \mathrm{N}\left(14\,000, \dfrac{9\,000\,000}{5}\right)$ ***(M1)***

$$\mathrm{P}(\bar{T} > 13\,000) = \mathrm{P}\left(Z > \frac{13\,000 - 14\,000}{\frac{3\,000}{\sqrt{5}}}\right) = \mathrm{P}\left(Z > -\frac{\sqrt{5}}{3}\right) = 0.772$$ ***(M1)(A1)***

[13 marks]

(ii) H_0 : Type and hours in stand by are independent.

H_1 : Type and hours in stand by are not independent. ***(A1)***

This is a χ^2 test of independence where the expected frequency for each cell is $\dfrac{\text{total of row} \times \text{total of column}}{\text{total frequency}}$. ***(A1)***

continued…

Question 5 (ii) continued

Expected frequencies

Type ⇒ / Hours in stand-by ⇓	A	B	C	D	E
$x \geq 180$	91.2	148.8	157.2	142.8	60
$150 \leq x < 180$	30.40	49.60	52.40	47.60	20
$120 \leq x < 150$	26.45	43.15	45.59	41.41	17.40
$x < 120$	3.95	6.45	6.81	6.19	2.60

(A2)

Since last row contains cells with less than 5 observations, we must combine it with the one above it. Expected are in italics. ***(R1)***

Type ⇒ / Hours in stand-by ⇓	A	B	C	D	E
$x \geq 180$	88/*91.2*	148/*148.8*	158/*157.2*	144/*142.8*	62/*60*
$150 \leq x < 180$	28/*30.40*	50/*49.60*	54/*52.40*	48/*47.60*	20/*20*
$x < 150$	36/*30.40*	50/*49.60*	50/*52.4*	46/*47.6*	18/*20*

$\chi^2 = \frac{\sum (f_o - f_e)^2}{f_e} = 1.841$, with 8 degrees of freedom, $\chi_c^2 = 15.507$. ***(M1)(A1)(A1)***

Since 1.841 < 15.507, we fail to reject the null hypothesis and conclude that we have no evidence to say that Stand-By time of cell phones is related to the type used. ***(R1)***

[9 marks]

Total [22 marks]