INTERNATIONAL BACCALAUREATE
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MARKSCHEME

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FURTHER MATHEMATICS

Standard Level

Paper 1

10 pages

1. (a) When performing a χ^2 test where ν, the number of degrees of freedom is 1, the continuity correction is used to compensate for the fact that the χ^2 distribution is continuous, whereas cell entries are discrete. ***(A1)***

(b) H_0 : coin is fair.
H_1 : coin is not fair.
Assuming H_0, the expected numbers of heads and tails are 100 each.

$$\chi^2 = \frac{\{|108-100|-0.5\}^2}{100} + \frac{\{|92-100|-0.5\}^2}{100} = 1.125$$ ***(M1)(A1)***

For $\nu = 1, \chi^2$ value at 1% level of significance is 6.635. ***(A1)***

Since $6.635 > 1.125$, we cannot reject the null hypothesis that the coin is fair. ***(R1)***

[5 marks]

2. Given $(G, \circ)$ is a group. Let x, y be any two elements of G.
Since $x \circ x = e, x^{-1} = x$ for all $x \in G$ ***(C1)***
Hence $(x \circ y)^{-1} = x \circ y$. Also $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$ ***(M1)***
So $y^{-1} \circ x^{-1} = x \circ y$ ***(M1)***
Also $y^{-1} \circ x^{-1} = y \circ x$ ***(A1)***
So $x \circ y = y \circ x$ for all $x, y \in G$, and the group is Abelian. ***(R1)***

[5 marks]

3. Let P_k be the profit in thousands of dollars at the end of the kth year. Then
$P_1 = 30, P_2 = 68, P_3 = 144$

$$P_{n+1} = 2P_n + 8$$ ***(M1)(A1)***

Hence $P_{n+1} = 2(2P_{n-1} + 8) + 8$ ***(M1)***

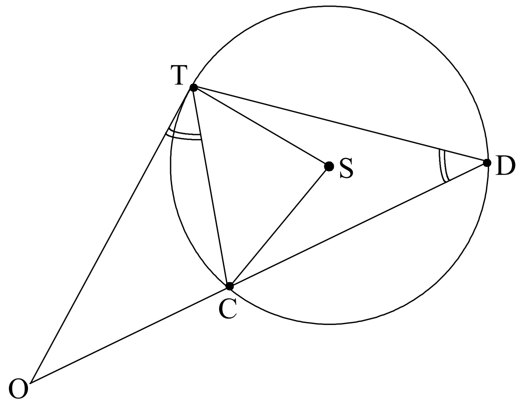
$$= 2^2(P_{n-1}) + 8(2+1)$$
$$= 2^3(P_{n-2}) + 8(2^2 + 2 + 1) = \ldots$$
$$= 2^n P_1 + 8\frac{2^{n-1}}{2-1} = 2^n P_1 + 2^n - 8 - 8$$
$$P_{n+1} = 2^n(P_1 + 8) - 8$$ ***(A1)***
$$P_n = 38000 \times 2^{n-1} - 8000$$ ***(A1)***

[5 marks]

4. We need to find a bijective function from $\mathbb{R}$ to $\mathbb{R}^+$ that preserves the operation, for example, let $f : \mathbb{R} \to \mathbb{R}^+$ be such that $f(x) = e^x$. ***(M1)***
$e^x = e^y$ implies $x = y$ and hence *f* is an injection. ***(M1)***
For any $c \in \mathbb{R}^+$, there exists $d \in \mathbb{R}$ such that $d = \ln c$ so that $f(d) = e^d = c$. So *f* is a surjection, and hence *f* is a bijection from $\mathbb{R}$ onto $\mathbb{R}^+$ ***(M1)***
Since, $f(x+y) = e^{x+y} = e^x \times e^y = f(x) \times f(y)$ ***(A1)***
f is an isomorphism. ***(R1)***

[5 marks]

5.



S is the centre of the circle.

$C\hat{D}T = \frac{1}{2} C\hat{S}T$ ***(M1)***

$O\hat{T}C = \frac{1}{2} C\hat{S}T$ ***(M1)***

$\Rightarrow O\hat{T}C = C\hat{D}T$ ***(A1)***

$T\hat{O}C = T\hat{O}D$

Therefore ΔOTC is similar to ΔODT ***(M1)***

$\frac{OT}{OC} = \frac{OD}{OT}$ ***(A1)***

$\Rightarrow OT^2 = OD \times OC$ ***(AG)***

[5 marks]

6. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^7} = \sum_{n=0}^{\infty} (-1)^n a_n$

$a_n = \frac{1}{(n+1)^7} \Rightarrow a_n \geq a_{n+1}, n = 0, 1, \ldots,$ ***(C1)***

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(n+1)^7} = 0$. ***(M1)***

Hence the series converges by the alternating series test. ***(R1)***

(b) For six decimal place accuracy, error E satisfies $|E| < |\text{first truncated term}|$

Therefore $\frac{1}{(n+1)^7} < 0.0000005$

Therefore $n+1 > 7.9 \Rightarrow n > 6.9$, so $n = 7$ (smallest integer value) ***(A1)***

Sum of the series correct to six decimal places is

$1 - \frac{1}{2^7} + \frac{1}{3^7} - \ldots + \frac{1}{7^7} = 0.992594$ (6 d.p.) ***(R1)***

OR

Since $8^{-7} = 0.000000476837$, ***(M1)***

The sum of the series correct to six decimal places is

$1 - \frac{1}{2^7} + \frac{1}{3^7} - \frac{1}{4^7} + \frac{1}{5^7} - \frac{1}{6^7} + \frac{1}{7^7} = 0.992594$ ***(A1)***

OR

Sum $= 0.992594$ ***(G2)***

[5 marks]

7. $f(x) = \mathrm{e}^{-x^2}$

$\int_0^5 \mathrm{e}^{-x^2}\,\mathrm{d}x$ may be approximated to 2 d.p. provided n satisfies

$\max\limits_{0<c<5}\left|f''(c)\right|\left(\frac{5-0}{12}\right)\left(\frac{5}{n}\right)^2 \le 5\times10^{-3}$ ***(M1)***

$\max\ \left|f''(c)\right| = 2$ ***(G1)***

$\Rightarrow 2\times\frac{5}{12}\times\frac{25}{n^2} \le 5\times10^{-3}$ ***(M1)***

$\Rightarrow n^2 \ge \frac{10\times25\times10^3}{12\times5}$

$\Rightarrow n^2 \ge \frac{25\times10^3}{6}$

$\Rightarrow n \ge 64.5$ ***(M1)***

thus $n = 65$ ***(A1)***

[5 marks]

8. If $y = mx + c$ is a tangent to the ellipse, then $(x, mx + c)$ will lie on the ellipse.

Hence $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$ ***(M1)***

or $x^2(b^2 + a^2m^2) + 2mxca^2 + (c^2 - b^2)a^2 = 0$ (1) ***(A1)***

If $y = mx + c$ is a tangent, then the equation (1) should have a double root and hence the discriminant should be zero.

Therefore $y = mx + c$ is a tangent to the ellipse if and only if (1) has two equal roots; ***(M1)***

if $4m^2c^2a^4 = 4(b^2 + a^2m^2)(c^2 - b^2)a^2$ ***(A1)***

$= 4b^2(c^2 - b^2)a^2 + 4a^4m^2(c^2 - b^2)$

It is true if (and only if) $c^2 - b^2 = a^2m^2$ ***(R1)***

$\Rightarrow c^2 = a^2m^2 + b^2$ ***(AG)***

[5 marks]

9. (a) Statistic: t statistic since sample sizes are small.
Variance: Pooled variance since population variances are equal. ***(A1)***

(b) $H_0 : \mu_1 = \mu_2$
$H_1 : \mu_1 \neq \mu_2$
Let $\bar{x}_1$ and $\bar{x}_2$ be the sample means for mines 1 and 2, respectively.
$\bar{x}_1 = 8230, \bar{x}_2 = 7940$ ***(G1)***

$$t = \frac{(8230 - 7940)}{\sqrt{\frac{(5 \times 112.25^2 + 6 \times 95.39^2)}{(5+6-2)}\left(\frac{1}{5} + \frac{1}{6}\right)}}$$ ***(M1)***

$= 4.19$ ***(A1)***

OR

2 sample t-test, $t = 4.19$, $p = 0.0023$, 9 degrees of freedom. ***(G2)***
$t_{.025}$ with $11 - 2 = 9$ degrees of freedom $= 2.262$.
Since calculated t value $4.19 > t_{.025} = 2.262$, we reject the null hypothesis. ***(R1)***

[5 marks]

10. If G has a spanning tree each vertex must be in that tree and hence G is connected. ***(M1)(R1)***
Conversely, if G is connected then using the breadth first search spanning tree algorithm to G, we get a set L of vertices and a set T of edges connecting vertices in L. ***(M1)***
Since T is a tree and G is connected each vertex of G is labelled. ***(M1)***
Thus L contains all the vertices of G and T is a spanning tree for the graph G. ***(R1)***

> **Note:** For the converse, some candidates may argue differently. Award marks as follows:
> Award ***(M1)(A1)*** for G is simple and connected then a tree can be created by taking one vertex and extending one edge and one vertex at a time to include all vertices without a cycle.
> Award ***(R1)*** for a tree is created which contains all the vertices, so we have a tree which spans G.

[5 marks]