Diploma Programme
Programme du diplôme
Programa del Diploma

Further mathematics
Higher level
Paper 2

Monday 26 October 2020 (morning)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 14]

(a) Use the Euclidean algorithm to show that the greatest common divisor of 411 and 339 is 3. [5]

(b) (i) Hence find the general solution of the Diophantine equation $411x - 339y = 3$.

(ii) Hence find the general solution of the linear congruence $137x \equiv 1 \pmod{113}$. [9]

2. [Maximum mark: 14]

(a) The matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \text{ where } a, b, c \in \mathbb{R}.$$

Verify that the inverse matrix of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$ [2]

(b) The set of matrices $\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\}$, where $a, b, c \in \mathbb{R}$, is denoted by S.

(i) Show that $\{S, *\}$ is a group, where $*$ denotes matrix multiplication. It may be assumed that matrix multiplication is associative.

(ii) Determine whether $\{S, *\}$ is Abelian.

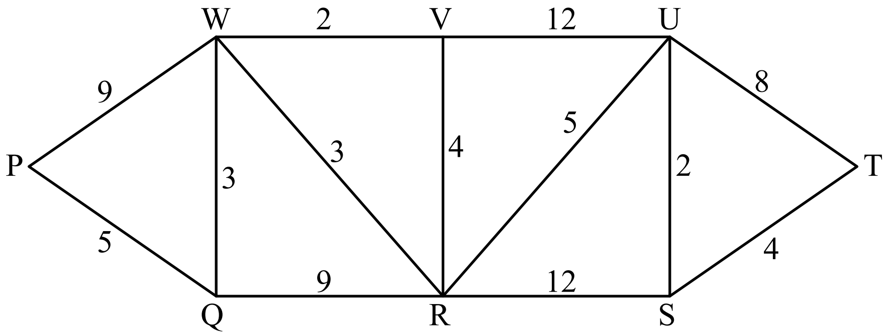
(iii) Identify any self-inverse elements of $\{S, *\}$. [11]

The set of matrices $\left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\}$, where $a, b, c \in \mathbb{Z}$, is denoted by T.

(c) Giving a reason, state whether $\{T, *\}$ is a subgroup of $\{S, *\}$. [1]

3. [Maximum mark: 14]

The following diagram shows the weighted graph G.



(a) Write down a Hamiltonian circuit in G starting at P. [1]

(b) (i) State what feature of G enables you to deduce that it is not bipartite.

(ii) Explain why G does not have an Eulerian trail.

(iii) An edge is removed from G, creating a new graph H, which does have an Eulerian trail. Identify all the possibilities for this edge. [5]

(c) (i) Use Dijkstra's algorithm in G to find the minimum weight path from P to T.

(ii) State the total weight of this path. [8]

4. [Maximum mark: 23]

The curve C has equation $y^2 = 4ax$ where $a > 0$. The distinct points $\mathrm{S}(as^2, 2as)$ and $\mathrm{T}(at^2, 2at)$ lie on C and move in such a way that $\mathrm{S\hat{O}T}$ is always a right angle, where O denotes the origin.

(a) Show that $st = -4$. [3]

(b) Show that the line (ST) passes through a fixed point on the x-axis independent of s and t. [6]

(c) Show that the tangents at the points S and T meet on the line $x + 4a = 0$. [6]

(d) (i) Determine the equation of the locus of the midpoint of $[\mathrm{ST}]$, showing it to be a parabola.

(ii) State the coordinates of the vertex and of the focus of this parabola. [8]

5. [Maximum mark: 13]

(a) Assuming the Maclaurin series for $\cos x$ and $\ln(1+x)$, show that the Maclaurin series for $\cos(\ln(1+x))$ is

$$1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{12}x^4+\dots.$$ [4]

(b) By differentiating the series in part (a), show that the Maclaurin series for $\sin(\ln(1+x))$ is $x-\frac{1}{2}x^2+\frac{1}{6}x^3+\dots$. [4]

(c) Hence determine the Maclaurin series for $\tan(\ln(1+x))$ as far as the term in x^3. [5]

6. [Maximum mark: 18]

(a) The relation R is defined such that aRb if and only if p divides a^2-b^2, where p is a fixed prime number greater than 2 and $a, b \in \mathbb{Z}^+$.

(i) Show that R is an equivalence relation.

(ii) Find, in terms of p, the two smallest positive integers that are related to 1, apart from 1 itself. [11]

(b) The relation S is defined such that xSy if and only if $1+|x-y|$ is the cube of a positive integer, where $x, y \in \mathbb{Z}^+$.

(i) Find the two smallest positive integers that are related to 2, apart from 2 itself.

(ii) Show that only two of the three requirements for S to be an equivalence relation are satisfied. [7]

7. [Maximum mark: 18]

The continuous random variable X has probability density function f given by

$$f(x) = \begin{cases} \dfrac{4x^3}{a^4} & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$$

where a is a positive constant. To estimate the value of a, a random sample of n observations is taken from the distribution of X. The sample mean is denoted by $\bar{X}$.

(a) Show that

$$\hat{a} = \frac{5}{4}\bar{X}$$

is an unbiased estimator for a. [5]

(b) Show that

$$\text{Var}(\hat{a}) = \frac{a^2}{24n}.$$ [7]

In a particular case, $n = 50$ and a is known to be 15.

(c) (i) Use the central limit theorem to determine, approximately, the interval with centre 15, in which $\hat{a}$ will lie with probability 0.95.

(ii) State whether it would be correct to conclude that your interval is an approximate 95% confidence interval for $\hat{a}$. Give a reason for your answer. [6]

8. [Maximum mark: 19]

Consider the matrix

$$A = \begin{bmatrix} 1 & \lambda & -2 \\ \lambda & 4 & 1 \\ 2 & 1 & \lambda \end{bmatrix}.$$

(a) Show that 3 is the only real value of λ for which A is singular. [4]

(b) Consider the system of equations

$$\begin{bmatrix} 1 & 3 & -2 \\ 3 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \mu \end{bmatrix}.$$

(i) Determine the value of μ for which the equations are consistent.

(ii) Solve the equations in that case. [6]

(c) Consider the system of equations

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 4 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ 5 \\ 1 \end{bmatrix}.$$

where a is an integer.

(i) Find the inverse of the matrix of coefficients, giving the answer in exact form.

(ii) Given that $y = x^2$, determine the value of a and hence the value of x and of y. [9]

9. [Maximum mark: 17]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-x}{y+x}$, where $x, y > 0$.

It is given that $y = 2$ when $x = 1$.

(a) Solve the differential equation, giving your answer in the form $f(x, y) = 0$. [9]

(b) The graph of y against x has a local maximum between $x = 2$ and $x = 3$. Determine the coordinates of this local maximum. [4]

(c) Show that there are no points of inflexion on the graph of y against x. [4]