Markscheme

May 2018

Further mathematics

Higher level

Paper 2

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking May 2018**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Misread

*If a candidate incorrectly copies information from the question, this is a misread (**MR**).*
*A candidate should be penalized only once for a particular misread. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$f'(x) = \left(2\cos(5x-3)\right)5 \ \left(=10\cos(5x-3)\right)$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:
Students must always use correct mathematical notation, not calculator notation.
Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

Note: In question 1, accept answers that round correctly to 2 significant figures.

1. (a) $aX + bY \sim \mathrm{N}\left(a\mu_1 + b\mu_2\ ,\ a^2\sigma_1^2 + b^2\sigma_2^2\right)$ ***A1A1***

Note: ***A1*** for N and the mean, ***A1*** for the variance.

[2 marks]

(b) (i) $X_1 + Y_1 \sim \mathrm{N}(5, 34)$ ***(A1)(A1)***

$\Rightarrow \mathrm{P}(X_1 + Y_1 < 11) = 0.848$ ***A1***

(ii) $3X_1 + 4Y_1 \sim \mathrm{N}(9+8\ ,\ 9\times 9 + 16\times 25)$ ***(A1)(M1)(A1)***

Note: Award ***(A1)*** for correct expectation, ***(M1)(A1)*** for correct variance.

$\sim \mathrm{N}(17, 481)$

$\Rightarrow \mathrm{P}(3X_1 + 4Y_1 > 15) = 0.536$ ***A1***

(iii) $X_1 + X_2 + Y_1 + Y_2 + Y_3 + Y_4 \sim \mathrm{N}(6+8,\ 2\times 9 + 4\times 25)$ ***(A1)(A1)***

$\sim \mathrm{N}(14, 118)$

$\Rightarrow \mathrm{P}(X_1 + X_2 + Y_1 + Y_2 + Y_3 + Y_4 < 30) = 0.930$ ***A1***

[10 marks]

(c) consider $\bar{X} - \bar{Y}$ ***(M1)***

$\mathrm{E}\left(\bar{X} - \bar{Y}\right) = 3 - 2 = 1$ ***A1***

$\mathrm{Var}\left(\bar{X} - \bar{Y}\right) = \frac{9}{2} + \frac{25}{4} (= 10.75)$ ***(M1)A1***

$\Rightarrow \mathrm{P}\left(\bar{X} - \bar{Y} > 0\right) = 0.620$ ***A1***

[5 marks]

Total [17 marks]

2. (a) Euler's method with step length $h = 0.1$ to find y when $x = 0.4$

x	y	$\frac{dy}{dx}$	$h\frac{dy}{dx}$	$y + h\frac{dy}{dx}$
0	2	5	0.5	2.5
0.1	2.5	4.2	0.42	2.92
0.2	2.92	3.7755	0.37755	3.29755
0.3	3.29755	3.49923	0.349923	3.64747
0.4	3.64747			

Note: Accept 3 significant figures in the table.

first line of table ***(M1)(A1)***
line 2 ***(A1)***
line 3 ***(A1)***
hence $y = 3.65$ ***A1***

Note: Accept any answer that rounds to 3.65.

[5 marks]

(b) (i) $$(5x+y)\frac{dy}{dx} = x + 5y$$

$$\left(5+\frac{dy}{dx}\right)\frac{dy}{dx} + (5x+y)\frac{d^2y}{dx^2} = 1 + 5\frac{dy}{dx}$$ ***M1A1A1***

Note: Award ***M1*** for a valid attempt to differentiate, ***A1*** for LHS, ***A1*** for RHS.

$$(5x+y)\frac{d^2y}{dx^2} = 1 + 5\frac{dy}{dx} - 5\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$$

$$(5x+y)\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2$$ ***AG***

(ii) $$(5x+y)\frac{d^2y}{dx^2} = 1 - \left(\frac{dy}{dx}\right)^2$$

$$\left(5+\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + (5x+y)\frac{d^3y}{dx^3} = -2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$ ***M1A1A1A1***

$$(5x+y)\frac{d^3y}{dx^3} = -2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 5\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$

$$(5x+y)\frac{d^3y}{dx^3} = -5\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$$ ***AG***

continued…

Question 2 continued

(iii) when $x=0$ $y=2$

when $x=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=5$ ***A1***

when $x=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-12$ ***A1***

when $x=0$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=120$ ***A1***

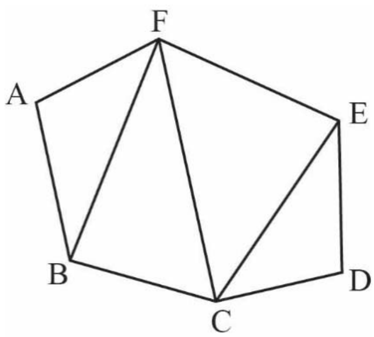
Note: Allow follow through from incorrect values of derivatives.

$y=2+5x-6x^2+20x^3$ ***M1A1***

[12 marks]

Total [17 marks]

3. (a)



A2

[2 marks]

(b) (i) two vertices are of odd degree ***A1***
to have an Eulerian circuit it must have all vertices of even degree ***R1***
hence no Eulerian circuit, but an Eulerian trail ***AG***

(ii) it allows Pauline to go through every door once (provided she starts in room B or room E) ***A1***
and she cannot return to the room in which she started ***A1***

[4 marks]

(c) (i) for example: $\mathrm{A}\rightarrow\mathrm{F}\rightarrow\mathrm{E}\rightarrow\mathrm{D}\rightarrow\mathrm{C}\rightarrow\mathrm{B}\rightarrow\mathrm{A}$ ***A2***

Note: Award ***A1*** if the cycle does not return to the start vertex.

(ii) she can visit every room once without repeating and return to the start ***A1***

[3 marks]

(d) $\mathrm{Z}\rightarrow\mathrm{V}\rightarrow\mathrm{Y}\rightarrow\mathrm{X}\rightarrow\mathrm{U}\rightarrow\mathrm{W}\rightarrow\mathrm{Z}$ ***A1***
$6+4+9+7+10+10=46$ ***A1***

[2 marks]

continued…

Question 3 continued

(e) attempt to find the minimal spanning tree ***(M1)***
VY
VW
UX
XY ***A2***

Note: Award ***A1*** if one error made.

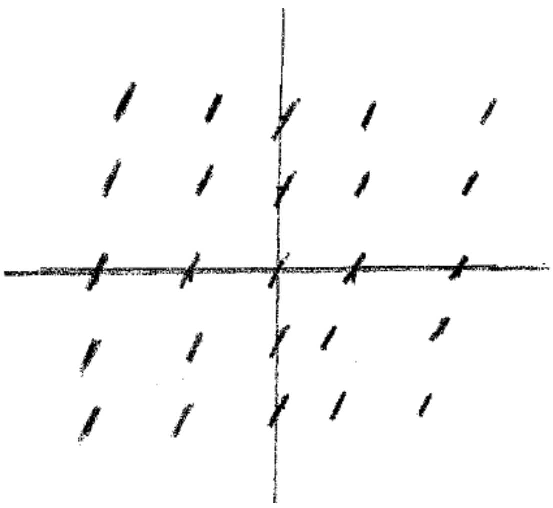
Note: Accept correct drawing of minimal spanning tree.

weight of minimal spanning tree $= 4+5+7+9=25$ ***(A1)***
since Z is removed, we add on VZ and ZY ***(M1)***
hence lower bound for route is $25+13=38$ ***A1***

[6 marks]

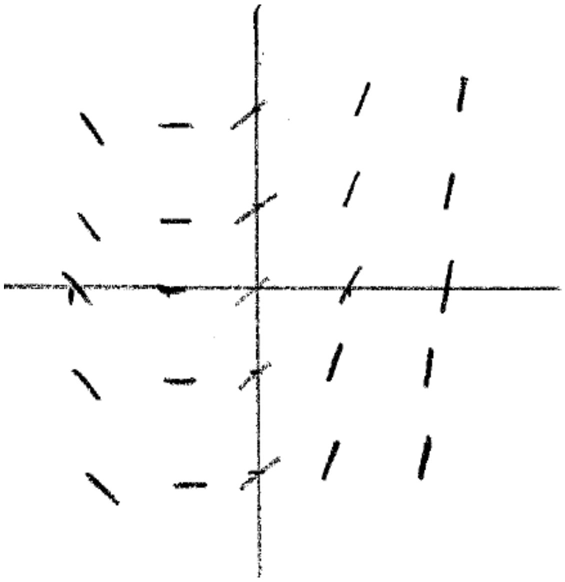
Total [17 marks]

4. (a) (i)



A2

(ii)

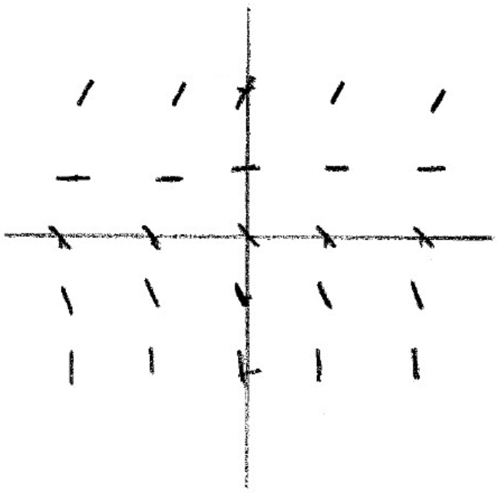


A2

continued…

Question 4 continued

(iii)



A2
[6 marks]

(b) (i) the slope is the same everywhere ***A1***

(ii) all points that have the same x coordinate have the same slope ***A1***
[2 marks]

(c) this is where a straight line appears on the slope field ***A1***
There is no other straight line, all the other solutions are curves ***A1***
[2 marks]

(d) given $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, the isoclines are $f(x, y) = k$ ***(M1)***

here the isoclines are $y = kx$ (or $x = ky$) ***(A1)***

any two differential equations of the correct form, for example
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ky}{x}, \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx}{y}, \frac{\mathrm{d}y}{\mathrm{d}x} = \sin\left(\frac{y}{x}\right), \frac{\mathrm{d}y}{\mathrm{d}x} = \sin\left(\frac{x}{y}\right)$ ***A1A1***

[4 marks]

Total [14 marks]

5. (a) $A = 4\int y\mathrm{d}x$ *(M1)*

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow$

$y = \frac{b\sqrt{a^2 - x^2}}{a}$ *(A1)*

let $x = a\cos\theta \Rightarrow y = b\sin\theta$ *M1*

$\frac{dx}{d\theta} = -a\sin\theta$ *A1*

when $x = 0$, $\theta = \frac{\pi}{2}$. When $x = a$, $\theta = 0$ *A1*

$\Rightarrow A = 4\int_{\frac{\pi}{2}}^{0} b\sin\theta\left(-a\sin\theta\right)\mathrm{d}\theta$ *M1*

$\Rightarrow A = -4ab\int_{\frac{\pi}{2}}^{0} \sin^2\theta\mathrm{d}\theta$

$\Rightarrow A = -2ab\int_{\frac{\pi}{2}}^{0} \left(1 - \cos 2\theta\right)\mathrm{d}\theta$ *M1*

$\Rightarrow A = -2ab\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}$ *A1*

$\Rightarrow A = -2ab\left[0 - 0 - \left(\frac{\pi}{2} - 0\right)\right]$ *M1*

$\Rightarrow A = \pi ab$ *AG*

[9 marks]

(b) (i) $b = 2$

hence $2\pi a = 8\pi \Rightarrow a = 4$ *A1*

hence major axis lies along the x-axis *A1*

(ii) $b^2 = a^2\left(1 - e^2\right)$ *(M1)*

$4 = 16\left(1 - e^2\right) \Rightarrow e = \frac{\sqrt{3}}{2}$ *A1*

(iii) coordinates of foci are $\left(\pm ae,\, 0\right) = \left(2\sqrt{3},\, 0\right), \left(-2\sqrt{3},\, 0\right)$ *A1A1*

(iv) equations of directrices are $x = \pm\frac{a}{e} = \frac{8}{\sqrt{3}}, -\frac{8}{\sqrt{3}}$ *A1A1*

[8 marks]

continued…

Question 5 continued

(c) $a=\frac{3}{2}, b=\frac{5}{2}$ ***(A1)***

hence equation is $\frac{4}{9}(x-2)^2+\frac{4}{25}(y-1)^2=1$ ***M1A1***

[3 marks]

Total [20 marks]

6. (a) $a+e+20=a(\bmod 100)$ ***(M1)***

$e=-20(\bmod 100)$ ***(A1)***

$e=80$ ***A1***

[3 marks]

(b) $a+a^{-1}+20=80(\bmod 100)$ ***(M1)***

inverse of a is $60-a(\bmod 100)$ ***A1***

[2 marks]

(c) 30 and 80 ***A1A1***

[2 marks]

(d) $a\circ(b*c)=a\circ(b+c+20)(\bmod 100)$

$=a+(b+c+20)-20(\bmod 100)$ ***(M1)***

$=a+b+c(\bmod 100)$ ***A1***

$(a\circ b)*(a\circ c)=(a+b-20)*(a+c-20)(\bmod 100)$ ***M1***

$=a+b-20+a+c-20+20(\bmod 100)$

$=2a+b+c-20(\bmod 100)$ ***A1***

hence we have shown that $a\circ(b*c)\neq(a\circ b)*(a\circ c)$ ***R1***

hence the operation $\circ$ is not distributive over $*$ ***AG***

Note: Accept a counterexample.

[5 marks]

(e) $\{0,5,10,15...\}$ ***A1***

$\{1,4,11,14...\}$ ***A1***

$\{2,3,12,13...\}$ ***A1***

$\{6,9,16,19...\}$ ***A1***

$\{7,8,17,18...\}$ ***A1***

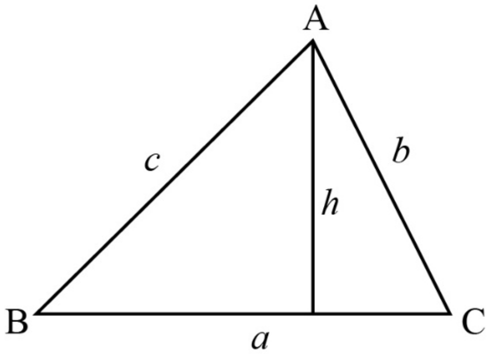
[5 marks]

(f) for example 10 and 50, 20 and 40, 0 and 60… ***A2***

[2 marks]

Total [19 marks]

7. (a) (i)



$\sin B = \frac{h}{c}$ and $\sin C = \frac{h}{b}$ ***M1A1***

hence $h = c\sin B = b\sin C$ ***A1***

by dropping a perpendicular from B, in exactly the same way we find

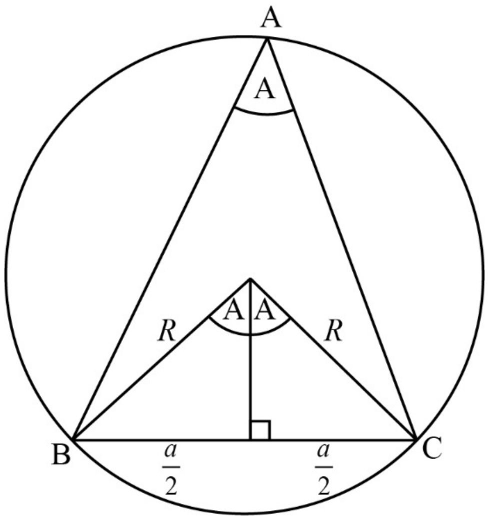
$c\sin A = a\sin C$ ***R1***

hence $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

(ii) area $= \frac{1}{2}ah$ ***M1A1***

$= \frac{1}{2}ab\sin C$ ***AG***

(iii)



since the angle at the centre of circle is twice the angle at the

circumference $\sin A = \frac{a}{2R}$ ***M1A1***

hence $\frac{a}{\sin A} = 2R$ and therefore $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ ***AG***

continued…

Question 7 continued

(iv) area of the triangle is $\frac{1}{2}ab\sin \mathrm{C}$ ***M1***

since $\sin \mathrm{C} = \frac{c}{2R}$ ***A1***

area of the triangle is $\frac{1}{2}ab\frac{c}{2R} = \frac{abc}{4R}$ ***AG***

[10 marks]

(b) (i) area of the triangle is $\frac{\pi R^2}{6}$ ***(M1)A1***

$(\mathrm{DE})^2 + (\mathrm{EF})^2 = 4R^2$ ***M1***

$(\mathrm{DE})^2 = 4R^2 - (\mathrm{EF})^2$

$\frac{1}{2}(\mathrm{DE})(\mathrm{EF}) = \frac{\pi R^2}{6} \Rightarrow (\mathrm{EF}) = \frac{\pi R^2}{3(\mathrm{DE})}$ ***M1A1***

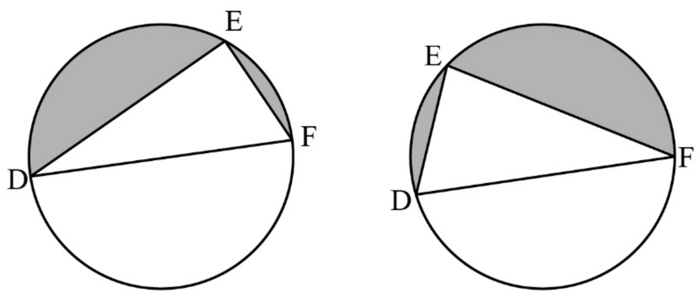
$(\mathrm{DE})^2 = 4R^2 - \frac{\pi^2 R^4}{9(\mathrm{DE})^2}$ ***A1***

$9(\mathrm{DE})^4 - 36(\mathrm{DE})^2 R^2 + \pi^2 R^4 = 0$ ***A1***

$(\mathrm{DE})^2 = \frac{36R^2 \pm \sqrt{1296R^4 - 36\pi^2 R^4}}{18}$ ***M1***

$(\mathrm{DE})^2 = \frac{36R^2 \pm 6R^2\sqrt{36-\pi^2}}{18} \left(= \frac{6R^2 \pm R^2\sqrt{36-\pi^2}}{3} \right)$ ***A1***

(ii)



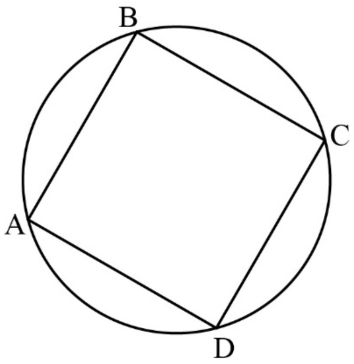
A1A1

[11 marks]

continued…

Question 7 continued

(c)



$\hat{A}+\hat{C}=180^\circ$ (cyclic quadrilateral) ***R1***

however $\hat{A}=\hat{C}$ (ABCD is a parallelogram) ***R1***

$\hat{A}=\hat{C}=90^\circ$ ***A1***

similarly $\hat{B}=\hat{D}=90^\circ$

hence ABCD is a rectangle ***AG***

[3 marks]

Total [24 marks]

8. (a) (i) $P(X=x)=pq^{x-1}$ for $x=1,2...$

$$G(t)=\sum_{x=1}^{\infty}t^x pq^{x-1}$$ ***M1***

$$=pt\sum_{x=1}^{\infty}(tq)^{x-1}$$ ***A1***

$$=pt(1+tq+(tq)^2...)$$ ***M1***

$$=\frac{pt}{1-tq}$$ ***AG***

(ii) $$G'(t)=\frac{(1-tq)p-pt(-q)}{(1-tq)^2}$$ ***M1A1***

$$E(X)=G'(1)$$ ***M1***

$$=\frac{(1-q)p+pq}{(1-q)^2}$$ ***A1***

$$=\frac{1}{p}$$ ***AG***

[7 marks]

continued…

Question 8 continued

(b) after 6 serves (3 serves each) we have $ABBAAB$

A serves	B serves		
3 wins	0 losses	$p_1 = {}^3C_3 p_A^3 q_A^0\, {}^3C_0 p_B^3 q_B^0$	***M1A1***
2 wins	1 loss	$p_2 = {}^3C_2 p_A^2 q_A^1\, {}^3C_1 p_B^2 q_B^1$	***A1***
1 win	2 losses	$p_3 = {}^3C_1 p_A^1 q_A^2\, {}^3C_2 p_B^1 q_B^2$	***A1***
0 wins	3 losses	$p_4 = {}^3C_0 p_A^0 q_A^3\, {}^3C_3 p_B^0 q_B^3$	***A1***

since ${}^3C_0 = {}^3C_3$, ${}^3C_1 = {}^3C_2$

$\sum_{x=0}^{x=3} \binom{3}{x}^2 (p_A)^x (p_B)^x (q_A)^{3-x} (q_B)^{3-x}$ ***AG***

[5 marks]

(c) for $N = 2$ serves are B, A respectively

$\mathrm{P}(N = 2) = \mathrm{P}(B \text{ wins twice}) + \mathrm{P}(A \text{ wins twice})$ ***(M1)***

$= 0.6 \times 0.3 + 0.4 \times 0.7$ ***A1***

$= 0.46$ ***A1***

[3 marks]

(d) for $M = \frac{1}{2}N$

$\mathrm{P}(M = 1) = \mathrm{P}(N = 2) = p_M$ ***M1***

$\mathrm{P}(M = 2) = \mathrm{P}(N = 4)$

$= \mathrm{P}\left(\begin{matrix}\text{game does not end after}\\ \text{first two serves}\end{matrix}\right) \times \mathrm{P}\left(\begin{matrix}\text{game ends after}\\ \text{next two serves}\end{matrix}\right) = (1 - p_M)\, p_M$ ***A1***

similarly $\mathrm{P}(M = 3) = (1 - p_M)^2\, p_M$ ***(A1)***

hence $\mathrm{P}(M = r) = (1 - p_M)^{r-1}\, p_M$ ***A1***

hence M has a geometric distribution ***AG***

$\mathrm{P}(M = 1) = \mathrm{P}(N = 2) = p_M = 0.46$ ***A1***

hence $\mathrm{E}(M) = \frac{1}{p} = \frac{1}{0.46} = 2.174$

$\mathrm{E}(N) = \mathrm{E}(2M) = 2\mathrm{E}(M)$ ***M1***

$= 4.35$ ***A1***

[7 marks]

Total [22 marks]