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Candidate session number			
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Examination session (May or November)	MAY	Year	2013

Diploma Programme subject in which this extended essay is registered: MATHEMATICS  
(For an extended essay in the area of languages, state the language and whether it is group 1 or group 2.)

Title of the extended essay: HOW CAN MATHEMATICS BE USED TO WORK OUT THE OPTIMAL DISTANCE FROM THE TRY LINE TO POSITION THE BALL FOR A CONVERSION KICK IN RUGBY UNION?

**Candidate's declaration**

*This declaration must be signed by the candidate; otherwise a grade may not be issued.*

The extended essay I am submitting is my own work (apart from guidance allowed by the International Baccalaureate).

I have acknowledged each use of the words, graphics or ideas of another person, whether written, oral or visual.

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Please comment, as appropriate, on the candidate's performance, the context in which the candidate undertook the research for the extended essay, any difficulties encountered and how these were overcome (see page 13 of the extended essay guide). The concluding interview (viva voce) may provide useful information. These comments can help the examiner award a level for criterion K (holistic judgment). Do not comment on any adverse personal circumstances that may have affected the candidate. If the amount of time spent with the candidate was zero, you must explain this, in particular how it was then possible to authenticate the essay as the candidate's own work. You may attach an additional sheet if there is insufficient space here.

THE CANDIDATE HAS RESEARCHED THE TOPIC THOROUGHLY AND PRODUCED A WELL-WRITTEN ESSAY ON A SUBJECT THAT HE IS PASSIONATE ABOUT. IN DISCUSSIONS ON THE MATHEMATICS WITHIN THE ESSAY, THE CANDIDATE HAS SHOWN A GOOD UNDERSTANDING OF THE DIFFERENT METHODS USED. HE HAS ALSO DEMONSTRATED INITIATIVE AND WAS ENTHUSIASTIC THROUGHOUT.

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I have read the final version of the extended essay that will be submitted to the examiner.

To the best of my knowledge, the extended essay is the authentic work of the candidate.

I spent  hours with the candidate discussing the progress of the extended essay.

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**Assessment form (for examiner use only)**

Criteria	Achievement level					
	Examiner 1	maximum	Examiner 2	maximum	Examiner 3	
A research question	<input type="text" value="2"/>	2	<input type="text"/>	2	<input type="text"/>	
B introduction	<input type="text" value="1"/>	2	<input type="text"/>	2	<input type="text"/>	
C investigation	<input type="text" value="4"/>	4	<input type="text"/>	4	<input type="text"/>	
D knowledge and understanding	<input type="text" value="4"/>	4	<input type="text"/>	4	<input type="text"/>	
E reasoned argument	<input type="text" value="4"/>	4	<input type="text"/>	4	<input type="text"/>	
F analysis and evaluation	<input type="text" value="3"/>	4	<input type="text"/>	4	<input type="text"/>	
G use of subject language	<input type="text" value="4"/>	4	<input type="text"/>	4	<input type="text"/>	
H conclusion	<input type="text" value="2"/>	2	<input type="text"/>	2	<input type="text"/>	
I formal presentation	<input type="text" value="3"/>	4	<input type="text"/>	4	<input type="text"/>	
J abstract	<input type="text" value="2"/>	2	<input type="text"/>	2	<input type="text"/>	
K holistic judgment	<input type="text" value="3"/>	4	<input type="text"/>	4	<input type="text"/>	
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Mathematics Extended Essay

How can mathematics be used to work out the optimal distance from the try line to position the ball for a conversion kick in rugby union?

**Abstract**

This investigation considers the research question: “How can mathematics be used to work out the optimal distance from the try line to position the ball for a conversion kick in rugby union?” This issue has a significant impact on the chance of success, and therefore a potentially great effect on the outcome of the match.

In undertaking the investigation, I research into the official pitch dimensions and functions, and IRB conversion rules. Together with a comparison with the Regiomontanus’ problem, I build upon this knowledge, considering the relevant variables and making assumptions to simplify the problem. I use this information to set up a model to allow for optimisation. My initial approach uses trigonometry and calculus to calculate the optimal distance from the try line and the maximum conversion angle at this point. I then employ an alternative approach, utilising circle theorems to find different equations which give the same results. After exploring the similarity of the asymptote to the hyperbola, I consider the need to clear the crossbar in addition to the posts, producing a modified hyperbola to find the improved horizontal distance.

In conclusion, the optimum distance back and maximum conversion angle are found to be  $y_{max} = \sqrt{ab}$  or  $y = \sqrt{x^2 - \left(\frac{d}{2}\right)^2}$ , and  $(\theta)_{max} = \tan^{-1}\left(\frac{(d)(y)}{y^2+ab}\right)$  or  $\theta = \sin^{-1}\left(\frac{d}{2x}\right)$  respectively. However, considering the crossbar, I state that kickers should aim to place the ball on the improved rectangular hyperbola,  $z = \sqrt{x^2 - \left(\frac{d}{2}\right)^2 - h^2}$ . Nevertheless, as the hyperbolae have similar asymptotes, if this is not possible the ball should be placed on the asymptote. However, its importance depends on the  $x$  value – when  $x$  is small, kickers should choose the point that allows greatest elevation; when  $x$  is large, kickers should choose a point on a  $45^\circ$  line from the posts, as the hyperbola is asymptotic. Nonetheless, there are many limitations that are considered for further investigation.

**Word Count:** 299

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## Introduction

This essay considers the question: “How can mathematics be used to work out the optimal distance from the try line to position the ball for a conversion kick in rugby union?” I aim to use mathematics to solve a significant real-life problem of interest to me: the optimisation of a conversion kick in rugby union. To achieve this, I will set up a model and use mathematics to find the best distance from the try line to position the ball for the conversion.

The mathematical analysis of such a realistic problem may appear to over-complicate a dilemma which, some would argue, has an intuitive answer – the majority of kickers seem to instinctively ‘know’, utilising intuition and experience, where to place the ball for a conversion. However, this does not mean that it is unworthy of investigation. Perhaps this causes less than optimal positioning of conversions? If this is the case, then could mathematical analysis of the problem increase their chances of success?

Furthermore, this is a significant problem in rugby union, as matches are often decided by a couple of points. With two points given for a successful conversion, the success or failure of a kick can greatly affect the outcome of a game; over many matches, these points could make the difference between promotion and relegation. Consequently, this problem is significant and worthy of investigation.

Several mathematicians have investigated this problem before, producing similar results but often developing on previous work. Hughes (1978)<sup>1</sup> seems the first to have investigated this scenario, suggesting that the locus of optimal points lies on a hyperbola. Avery (1989)<sup>2</sup> later produced the same results using a different method, whilst Worsnup (1989)<sup>3</sup> developed Hughes’ findings, suggesting the use of the asymptote of the hyperbola. De Villiers (1999)<sup>4</sup> later explored the same problem, accounting for the clearing of the crossbar. Eastaway and Wyndham (1998)<sup>5</sup>, and later Eastaway and Haigh (2007)<sup>6</sup>, both touched on the problem in their respective books, before Polster and Ross (2010)<sup>7</sup> collated these ideas and developed them themselves. The problem has therefore been investigated several times, but, due to the differences in methods, further investigation of the problem seems justified.

The personal motivation for me in pursuing this investigation is that I have an individual interest too in that I have both played rugby and studied maths throughout secondary education.

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<sup>1</sup> Anthony Hughes (1978), *Conversion Attempts in Rugby Football*, from *The Mathematical Gazette, A Journal of the Mathematical Association*, Vol. 62, No. 422 (Dec. 1978), pp. 292-293, The Mathematical Association

<sup>2</sup> Peter Avery (1989), *Mathematics in Sport*, from *The Mathematical Gazette, A Journal of the Mathematical Association*, Vol. 73, No. 463 (Mar. 1989), pp. 1-6, The Mathematical Association

<sup>3</sup> G. Worsnup (1989), *An Aid to Conversions in Rugby*, from *The Mathematical Gazette, A Journal of the Mathematical Association*, Vol. 73, No. 465 (Oct. 1989), pp. 225-226, The Mathematical Association

<sup>4</sup> Michael de Villiers (1999), *Place Kicking Locus in Rugby*, published in *Pythagoras*, 1999, No. 49, Aug, 64-67, Official Journal of AMESA, <http://mzone.mweb.co.za/residents/profmd/rugby.pdf>, (accessed 20/11/12)

<sup>5</sup> Rob Eastaway and Jeremy Wyndham, foreword by Tim Rice (1998), *Why do Buses Come in Threes? The hidden mathematics of everyday life*, pp. 57-58, Robson Books

<sup>6</sup> Rob Eastaway and John Haigh (2007), *Beating the Odds: the hidden mathematics of sport*, pp. 117-118, Robson Books

<sup>7</sup> Burkard Polster and Marty Ross (2010), *Mathematical Rugby*, from *The Mathematical Gazette, A Journal of the Mathematical Association*, Vol. 94, No. 531 (Nov. 2010), pp. 450-463, The Mathematical Association

This problem therefore allows me to explore two contrasting areas of interest, aiming to further my knowledge in both whilst producing a significant and useful solution to a real-life problem.

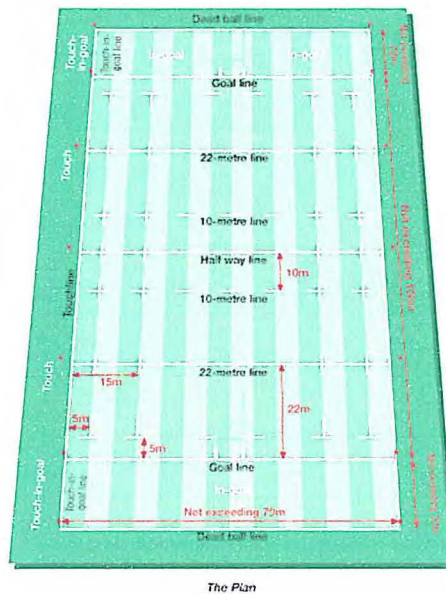
In summary, this question is worthy of investigation as it has a significant impact on the likelihood of success of the conversion kick, and, therefore, a potentially huge effect on the outcome of the match.



**Investigation**

To better understand the problem, its context must be considered. The primary source in gaining this background knowledge is the International Rugby Board (IRB), the governing body and law-makers of the game.

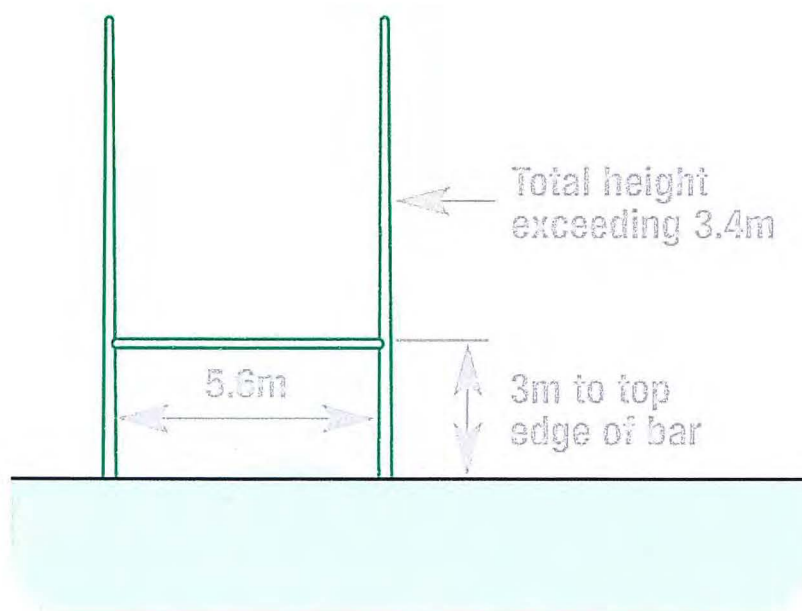
Firstly, we need to consider the dimensions and functions of the rugby pitch. The following official IRB diagrams illustrate this, focusing on the posts, an important component of this investigation:



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**Figure 1:** Plan view of an official rugby pitch, illustrating the names and dimensions of its constituent parts

<sup>8</sup> IRB (2012), *Laws of the Game Rugby Union*, [http://www.irblaws.com/EN/laws/2/1/2/before-the-match/the-ground/the-plan/#clause\\_2](http://www.irblaws.com/EN/laws/2/1/2/before-the-match/the-ground/the-plan/#clause_2) (accessed 20/11/12)



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**Figure 2:** Diagram showing the official dimensions of rugby posts

The rules relevant to conversion kicks can now be better understood. Law 9 of the IRB rule book, “Method of Scoring”<sup>10</sup>, highlights the methods of scoring in rugby union – the two relevant in this investigation are the try and the conversion.

A try is defined by the IRB Law 9.A.1 Points Values<sup>11</sup> as “when an attacking player is first to ground the ball in the opponents’ in-goal...” “When a player scores a try it gives the players’ team the right to attempt to score a goal by taking a kick at goal...This kick is a conversion kick...”, potentially adding an extra two points to the five for the try. A successful kick is one which passes through the posts and over the crossbar.

However, the attacking player cannot freely decide where to place the ball. The IRB Law 9.B.1 Taking a Conversion Kick (b)<sup>12</sup>, states: “The kick is taken on a line through the place where the try was scored.”

Consequently, the conversion kick can vary in difficulty depending on the position of the try, *i.e. the line perpendicular to the try line on which the conversion kick must be taken.*

<sup>9</sup> IRB (2012), *Laws of the Game Rugby Union*, [http://www.irblaws.com/EN/laws/2/1/6/before-the-match/the-ground/dimensions-for-goal-posts-and-crossbar/#clause\\_6](http://www.irblaws.com/EN/laws/2/1/6/before-the-match/the-ground/dimensions-for-goal-posts-and-crossbar/#clause_6) (accessed 20/11/12)

<sup>10</sup> IRB (2012), *Laws of the Game Rugby Union*, [http://www.irblaws.com/downloads/EN/IRB\\_Laws\\_2012\\_EN.pdf](http://www.irblaws.com/downloads/EN/IRB_Laws_2012_EN.pdf) (accessed 20/11/12)

<sup>11</sup> IRB (2012), *Op. Cit.*, p. 58, [http://www.irblaws.com/downloads/EN/IRB\\_Laws\\_2012\\_EN.pdf](http://www.irblaws.com/downloads/EN/IRB_Laws_2012_EN.pdf) (accessed 20/11/12)

<sup>12</sup> IRB (2012), *Op. Cit.*, (*Law 9.B.1 Taking a Conversion Kick (b)*), p. 59, [http://www.irblaws.com/downloads/EN/IRB\\_Laws\\_2012\\_EN.pdf](http://www.irblaws.com/downloads/EN/IRB_Laws_2012_EN.pdf) (accessed 20/11/12)

However, whilst the conversion line is constant, the kicker is permitted to choose the distance back from the try line at which he kicks the ball, and this is what the investigation seeks to explore, i.e. how can mathematics be used to optimise the variable over which they do have control – the distance back from the try line?

This problem produces a dilemma – if the kicker moves the ball back from the try line, the angle at which to successfully kick the ball through the posts increases, but too far back and he will not be able to clear the posts. Therefore, a compromise must be found between maximising the angle and minimising the distance, ensuring that the kicker is within range.

Research into this problem suggests, however, that this is not a unique problem; rather it is a specific example of the Regiomontanus' problem, a question asked by Johannes Muller in 1471<sup>13</sup>: "At what horizontal distance from an elevated rod would a person have to stand such that the appearance of the rod should be a maximum?"<sup>14</sup> This is therefore an angle maximisation problem, aiming to make the rod appear as large as possible. The conversion kick problem is therefore related to this question: the elevated rod and position of the person are replaced by the distance between the posts and position of the ball respectively, aiming to maximise the conversion angle and therefore maximise the appearance of the distance between the posts.

For the established problem to be analysed effectively, we need to consider the variables involved. For any particular conversion kick, the conversion line is fixed, but a number of variables could affect the optimal distance of the ball from the posts.

These factors relate to both the ball and kicker (e.g. path, spin and range) as well as external conditions (e.g. wind speed and direction). Therefore, assumptions need to be made to negate their effect on the outcome of the investigation:

Firstly, in terms of the ball and kicker, the ball will be modelled as a particle, to eliminate any variations in optimal position due to its size, shape or mass, etc. In addition, the path of the ball in flight will be assumed to follow a single vertical plane. Kickers tend to kick the ball in such a way that it follows a curved path, but this would affect the maximum angle and the optimal distance, and so it must be assumed to be straight. Moreover, kickers tend to apply spin to the ball, which may also affect its flight and therefore the optimum angle and distance, and so it must be assumed that the spin of the ball (and therefore its effect) is negligible. Furthermore, the kicking strength, range and accuracy of the kicker can vary greatly, resulting in a given distance being appropriate for a certain kicker but not another. This would cause the results to only be applicable to a certain proportion of kickers, so the kicking strength, range and accuracy must be assumed to be constant.

Secondly, external conditions can play a major role in the optimum position; e.g. windy conditions can either aid or hinder kicking, reducing and increasing required distance respectively. As a result, air resistance (and therefore wind) must be assumed to be negligible.

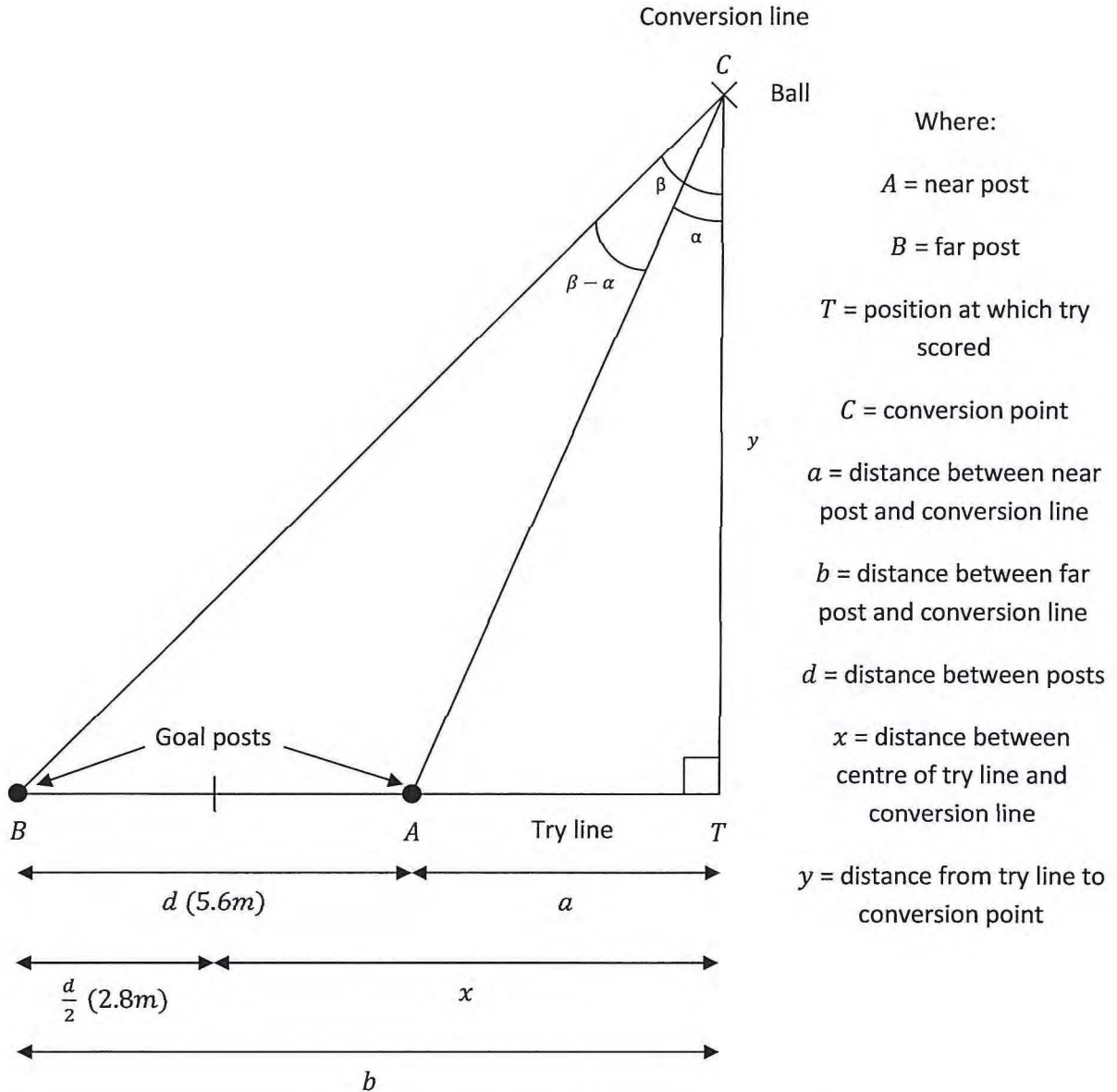
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<sup>13</sup> Kin Keung Poon and Hang-Chi Wong (2011), *Problem solving through an optimization problem in geometry*, Teaching Mathematics and its Applications, published by Oxford University Press on behalf of The Institute of Mathematics and its Applications, <http://teamat.oxfordjournals.org> (accessed 03/11/12)

<sup>14</sup> Mathematics2 (2011), *Calculus: Regiomontanus Angle Maximization Problem*, <http://mathematics2.com/Calculus/RegiomontanusAngleMaximisationProblem> (accessed 03/11/12)

The first response when solving the problem is to consider what information is known and what needs to be found. For any conversion, the distance between the posts is constant and the distance along the try line is fixed, whilst the distance back from the try line to the ball and the angle the ball makes with the posts need to be found.

This information can be used to model the problem:



**Figure 3:** Model of the problem on the rugby pitch

One approach is to use trigonometry to maximise the conversion angle formed by the conversion point,  $C$ , with the posts,  $A$  and  $B$ . We need to use trigonometry to find the maximum value of the tangent of this angle, before using calculus to optimise this function to find the optimum distance.

To optimise the distance,  $y$ , we need to use a function involving the tangent of the angle,  $(\beta - \alpha)$ , that separates the goal posts as viewed from the conversion point:

$$f(y) = \tan(\beta - \alpha)$$

This is a compound angle formula and so, using the formula<sup>15</sup>, we can rewrite this function:

$$f(y) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\tan\beta}$$

Returning to the model, right-angled trigonometry ( $\tan\theta = \frac{o}{a}$ ) can be used to rewrite the tangent  $\alpha$  and of  $\beta$ :

$$\tan\alpha = \frac{a}{y}$$

$$\tan\beta = \frac{b}{y}$$

Substituting these expressions into the equation, the function can be rewritten:

$$f(y) = \frac{\frac{b}{y} - \frac{a}{y}}{1 + \left(\frac{b}{y} \times \frac{a}{y}\right)}$$

Simplifying:

$$f(y) = \frac{(b - a)y}{y^2 + ab}$$

To optimise this function to find the optimum value of  $y$ , we need to differentiate. As the function is in the form  $y = \frac{u}{v}$ , we use the quotient rule<sup>16</sup>:

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

For this function,  $u = (b - a)y$  and  $v = y^2 + ab$ ; substituting these expressions into the equation:

$$f'(y) = \frac{(y^2 + ab)(b - a) - (b - a)y(2y)}{(y^2 + ab)^2}$$

Simplifying:

<sup>15</sup> IBO (2006), *Diploma Programme: Mathematics HL, Further Mathematics SL Information Booklet*, page 3

<sup>16</sup> IBO (2006), *Op. Cit.*, page 7

$$f'(y) = \frac{(b-a)(ab-y^2)}{(ab+y^2)^2}$$

To find the optimum value for  $y$ , we set the equation equal to 0:

$$y^2 = ab$$

To find the maximum value for  $y$ , we take the square root of both sides (giving only a positive result as length is positive):

$$y_{max} = +\sqrt{ab}$$

The maximum angle at this position now needs to be found, by substituting the optimal value of  $y$  into the inverse tangent function:

$$(\beta - \alpha)_{max} = \tan^{-1}\left(\frac{(b-a)y}{y^2 + ab}\right)$$

Since the distance  $(b-a)$  is equal to the distance between the posts,  $d$ , this can be simplified:

$$(\beta - \alpha)_{max} = \tan^{-1}\left(\frac{(d)(y)}{y^2 + ab}\right)$$

We now need to add values into the model to observe and investigate the results and any patterns. For example, when a try is scored on the touchline, the conversion kick can be taken at any point back along the line. Since the width of the pitch is 70m for this investigation,  $x$  (the distance from the centre of the pitch to the touchline) is therefore 35m. The model illustrates that  $a = x - \frac{d}{2}$  and  $b = x + \frac{d}{2}$ , and, since  $d$  is set at 5.6m,  $a$  and  $b$  can be calculated:

$$a = x - \frac{d}{2} = 35 - \frac{5.6}{2} = 35 - 2.8 = 32.2m$$

$$b = x + \frac{d}{2} = 35 + \frac{5.6}{2} = 35 + 2.8 = 37.8m$$

These value of  $a$  and  $b$  can be substituted into the previous equation to find the optimal  $y$ -value at this position:

$$y_{max} = \sqrt{ab}$$

$$y_{max} = \sqrt{32.2 \times 37.8}$$

$$y_{max} = 34.88782022 = 34.9m \text{ (3s. f.)}$$

These values can be substituted into the previous equation to find the maximum angle at this point:

$$(\beta - \alpha)_{max} = \tan^{-1}\left(\frac{(d)(y)}{y^2 + ab}\right)$$

$$(\beta - \alpha)_{max} = \tan^{-1} \left( \frac{(5.6)(34.88782022)}{(34.88782022)^2 + (32.2 \times 37.8)} \right)$$

$$(\beta - \alpha)_{max} = 4.588565736 = 4.59^\circ \text{ (3s.f.)}$$

Therefore, if the try is scored on the touchline, the farthest point from the centre of the posts, the optimal distance back at which to kick is 34.9m (3s.f.), creating a maximum conversion angle of 4.59° (3s.f.).

Considering the problem visually, if the try is scored closer along the try line to the goalposts, the values of  $a$  and  $b$  would be smaller, suggesting that the optimum value of  $y$  would decrease whilst the maximum conversion angle would increase.

Nevertheless, we must demonstrate this numerically; using the calculations shown previously, we can create a table of values to illustrate the outcomes of various distances of a conversion along the try line:

Distance $x$ (along try line from centre) / $m$ (3s.f.)	Optimum distance $y$ (perpendicular to try line) / $m$ (3s.f.)	Maximum conversion angle $\theta$ / ° (3s.f.)
0	0	180
5.00	4.14	34.1
10.0	9.60	16.3
15.0	14.7	10.8
20.0	19.8	8.05
25.0	24.8	6.43
30.0	29.9	5.36
35.0	34.9	4.59

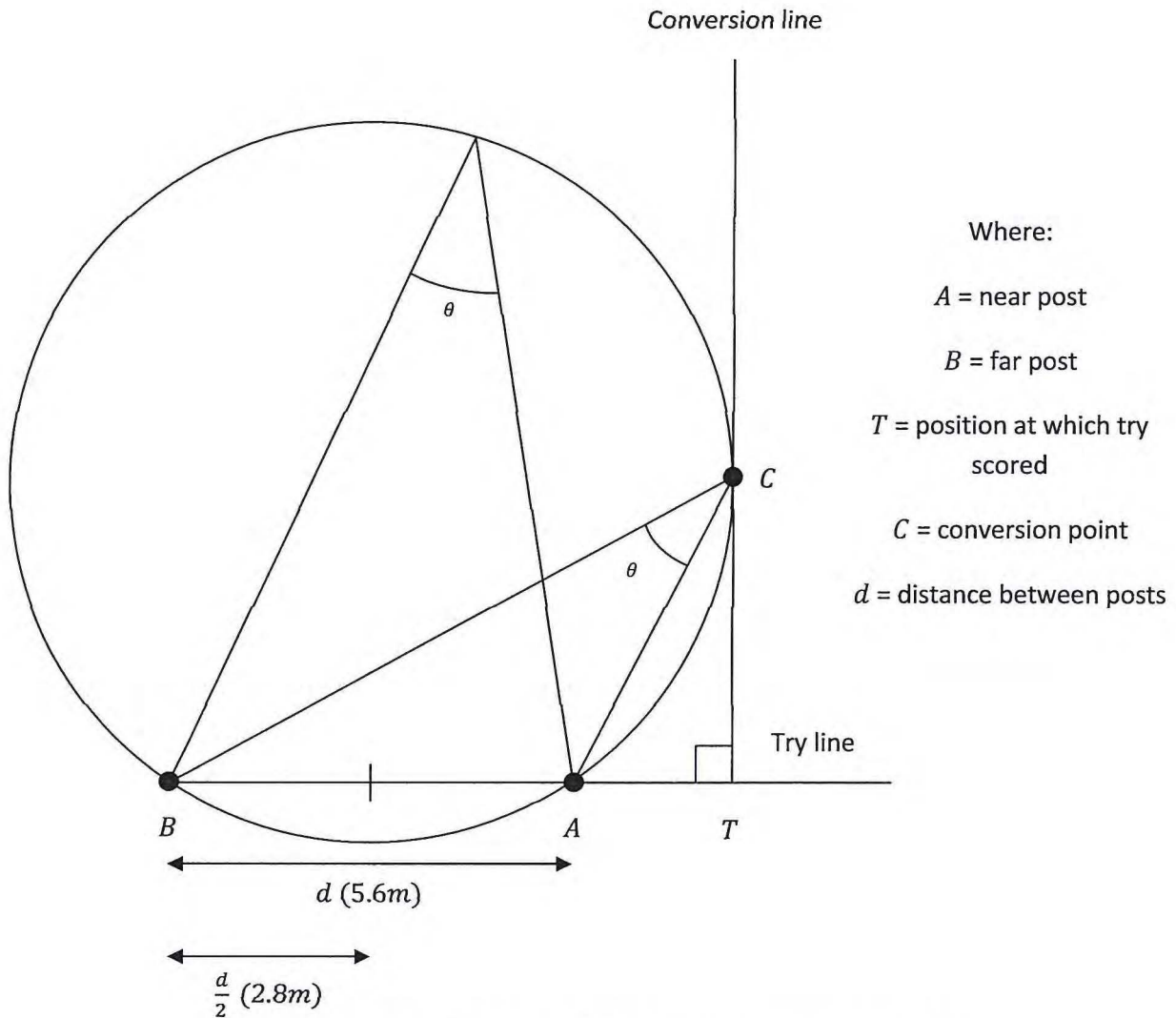
These results are supported by Avery<sup>17</sup> and Fortin<sup>18</sup>, albeit with different labelling and slightly differing methods, suggesting that my results are reliable.

The equations for the optimal distance and maximum angle therefore seem to produce valid results when tested, agreeing with our visualisation of the problem – as the distance along the try line from the centre increases, the optimal distance from the try line increases whilst the maximum conversion angle decreases.

<sup>17</sup> Peter Avery (1989), Op. Cit.

<sup>18</sup> Philippe Fortin (2007), *TI-nspire: Playing Rugby*, ©2007 Texas Instruments Incorporated, [http://compasstech.com.au/TNSINTRO/TI-NspireCD/Exemplary\\_Activities\\_PDF/Act9\\_PlayingRugby.pdf](http://compasstech.com.au/TNSINTRO/TI-NspireCD/Exemplary_Activities_PDF/Act9_PlayingRugby.pdf), (accessed 03/11/12)

However, confirming this using an alternative method would determine reliability - Eastaway and Wyndham<sup>19</sup> suggest that, if a circle is created which passes through the posts and touches the conversion line, “the point where the circle touches the line which is the tangent, is the best place to kick from,” as illustrated:



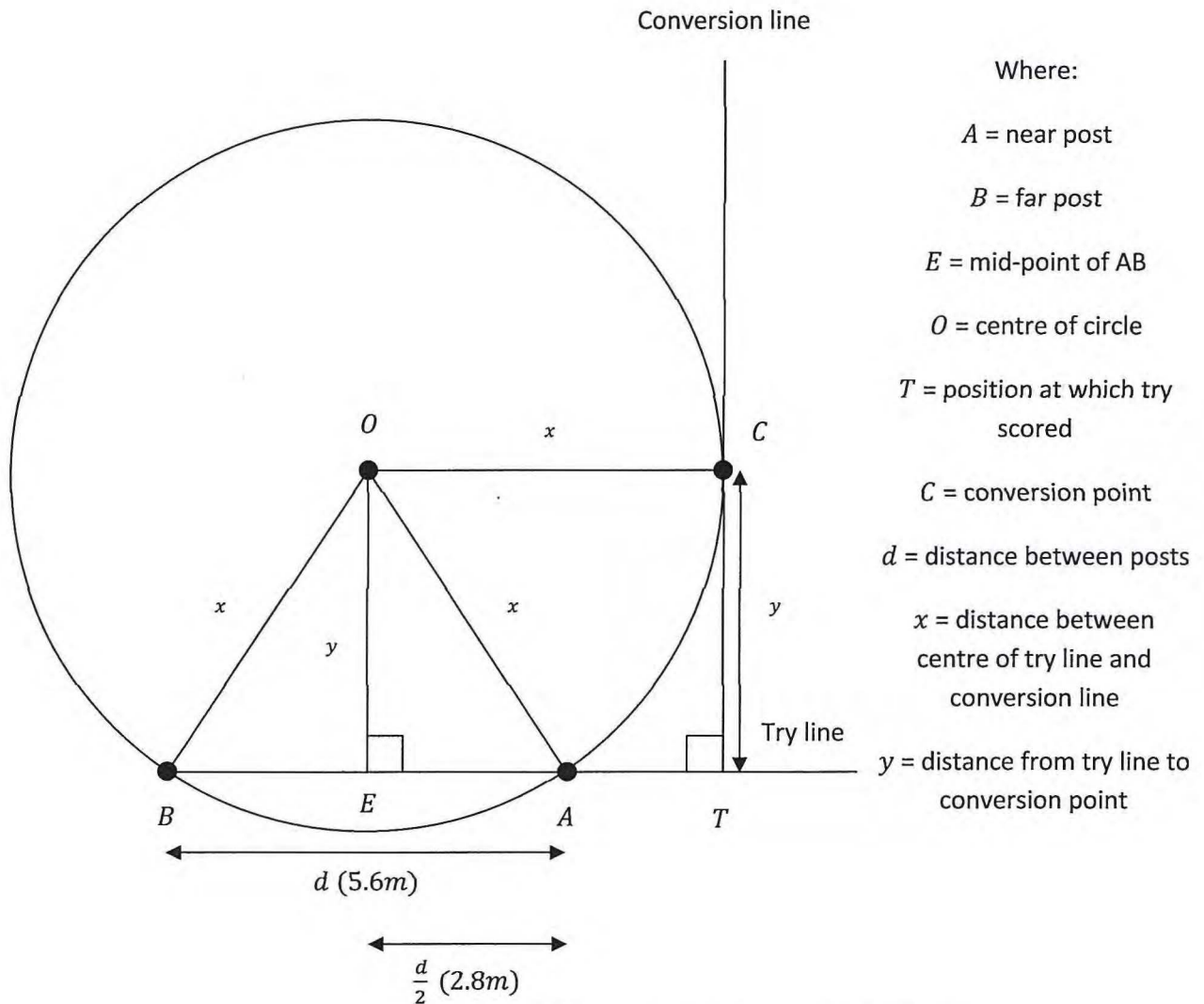
**Figure 4:** Diagram to show angles in same segment circle theorem in relation to problem

Using circle theorems, any angle within the circle which subtends the posts will be equal. But since the conversion must be along the conversion line, the only relevant point is where it is tangent to the circle. Closer and the angle is smaller and so gives the kicker less area to aim at; further away and the kicker has to kick it further than necessary, decreasing the chance of success. Polster and Ross<sup>20</sup> develop this approach, as illustrated:

<sup>19</sup> Rob Eastaway and Jeremy Wyndham, foreword by Tim Rice (1998), Op. Cit.

<sup>20</sup> Burkard Polster and Marty Ross (2010), Op. Cit.





**Figure 5:** Model of the problem on the rugby pitch using a combination of circle theorem methods

Utilising the right-angled triangle  $AEO$ , Pythagoras' Theorem can be used to find  $y$ :

$$x^2 = y^2 + \left(\frac{d}{2}\right)^2$$

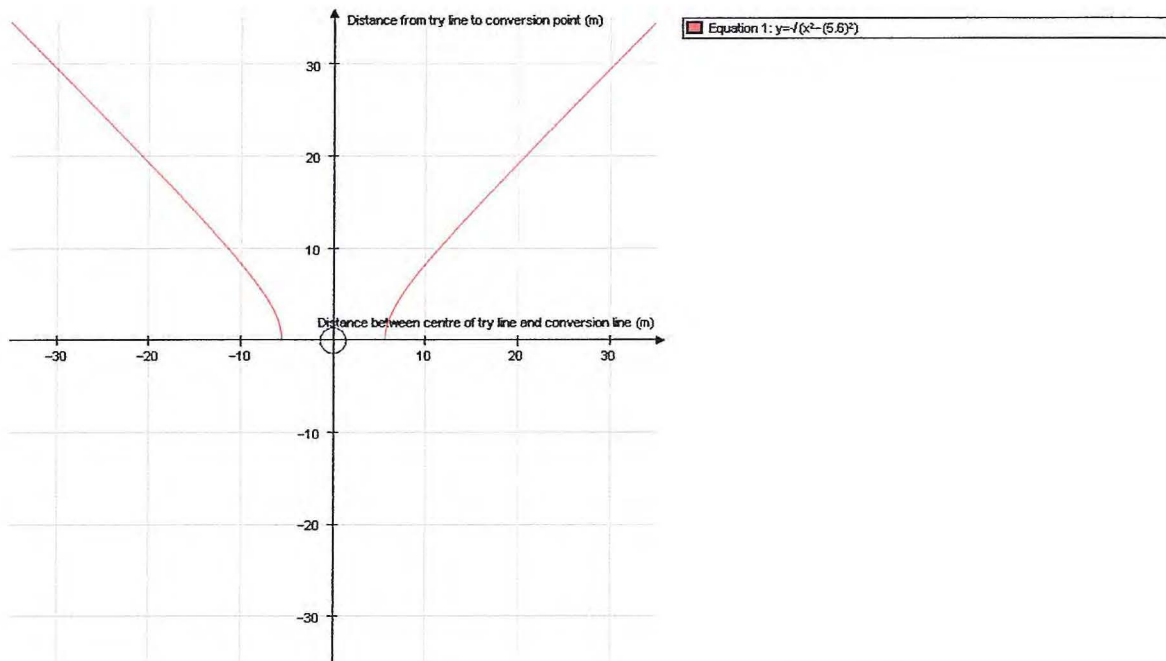
Rearranging:

$$y = \sqrt{x^2 - \left(\frac{d}{2}\right)^2}$$

This equation is a right-angled hyperbola, so all the optimal points lie on this line, as supported by Hughes<sup>21</sup>: "The locus of points from which to take the conversions is thus half of each of the branches of a rectangular hyperbola..."

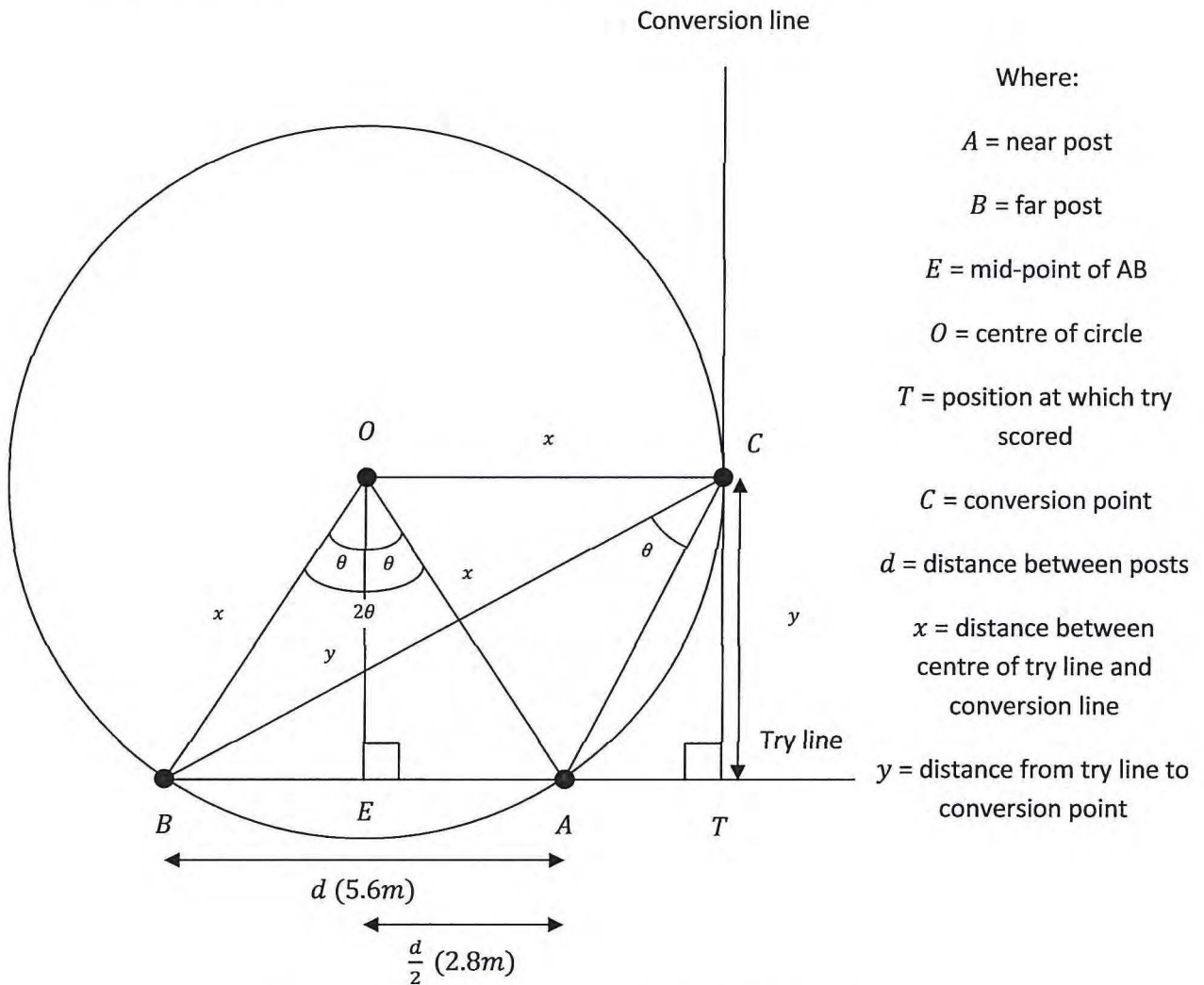
<sup>21</sup> Anthony Hughes (1978), Op. Cit.

To illustrate this visually, technology can be used (Autograph):



**Figure 6:** Graph showing the hyperbola  $y = \sqrt{x^2 - \left(\frac{5.6}{2}\right)^2}$

To calculate the maximum conversion angle too, we can redraw the model:



**Figure 7:** Model of problem on the rugby pitch highlighting the angles involved.

Letting angle ACB be  $\theta$ , the angle AOB is  $2\theta$ , since the circle theorem states that the angle at the centre is twice the angle at the circumference. However, if the triangle AOB is bisected, the angle AOE is therefore  $\theta$ . We can now use right-angled trigonometry to find  $\theta$ . Since we know the opposite and hypotenuse,  $x$  and  $\frac{d}{2}$  respectively, and, by trigonometry,  $\sin \theta = \frac{o}{h}$ , the sine function is:

$$\sin \theta = \frac{o}{h} = \frac{\left(\frac{d}{2}\right)}{x} = \frac{d}{2x}$$

To find  $\theta$ , we take the inverse of this function:

$$\theta = \sin^{-1}\left(\frac{d}{2x}\right)$$

We now need to add values into the model in order to compare the results, to determine reliability. Using the touchline example again, where  $x = 35m$ , and  $d = 5.6m$ ,  $y$  can be calculated:

$$y = \sqrt{x^2 - \left(\frac{d}{2}\right)^2}$$

$$y = \sqrt{35^2 - \left(\frac{5.6}{2}\right)^2}$$

$$y = 34.88782022 = 34.9m \text{ (3s.f.)}$$

Likewise, the same values can be used to find  $\theta$ :

$$\theta = \sin^{-1}\left(\frac{d}{2x}\right)$$

$$\theta = \sin^{-1}\left(\frac{5.6}{2 \times 35}\right)$$

$$\theta = 4.588565736 = 4.59^\circ \text{ (3s.f.)}$$

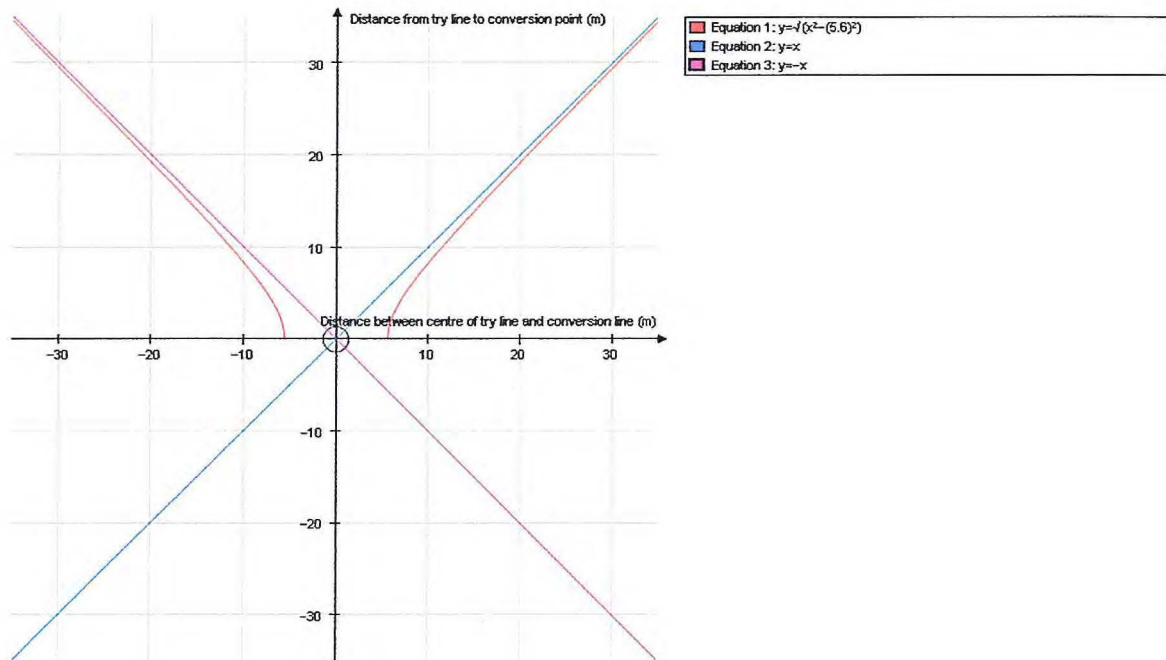
Therefore, from the touchline, the optimal  $y$ -value is  $34.9m$  (3s.f.), creating a maximum  $\theta$  value of  $4.59^\circ$  (3s.f.).

We now need to create a table of values to demonstrate the results at various  $x$ -values:

Distance $x$ (along try line from centre) / $m$ (3s.f.)	Optimum distance $y$ (perpendicular to try line) / $m$ (3s.f.)	Maximum conversion angle $\theta$ / $^\circ$ (3s.f.)
0	0	180
5.00	4.14	34.1
10.0	9.60	16.3
15.0	14.7	10.8
20.0	19.8	8.05
25.0	24.8	6.43
30.0	29.9	5.36
35.0	34.9	4.59

These results are identical to those achieved previously, suggesting reliability. Nevertheless, there are limitations to both methods, as discussed later.

However, returning to the rectangular hyperbola, it appears similar to the graph of  $y = x$ , particularly at larger values of  $x$ , as pointed out by Worsnup<sup>22</sup>, who highlights that this is because  $y = x$  is an asymptote to the hyperbola. This can be illustrated using technology (Autograph):



**Figure 8:** Graph showing the hyperbola  $y = \sqrt{x^2 - \left(\frac{5.6}{2}\right)^2}$  and its asymptotes  $y = x$  and  $y = -x$

He asks what difference it would make if the conversion point was chosen on the asymptote  $y = x$  rather than the hyperbola<sup>23</sup>.

We therefore need to create another table of values to include the optimal distance and maximum conversion angle when the points are on the asymptote  $y = x$  rather than the original hyperbola:

<sup>22</sup> G. Worsnup (1989), Op. Cit.

<sup>23</sup> Ibid.

Distance $x$ (along try line from centre) / $m$ (3s.f.)	Optimum distance $y$ (perpendicular to try line when on hyperbola) / $m$ (3s.f.)	Optimum distance $y'$ (perpendicular to try line when on asymptote) / $m$ (3s.f.)	Percentage difference between $y$ and $y'$ / % (3s.f.)	Maximum conversion angle $\theta$ (when on hyperbola) / $^\circ$ (3s.f.)	Maximum conversion angle $\theta'$ (when on asymptote) / $^\circ$ (3s.f.)	Percentage difference between $\theta$ and $\theta'$ / $^\circ$ (3s.f.)
0	0	0	0	180	180	0
5.00	4.14	5.00	20.8	34.1	33.6	1.47
10.0	9.60	10.0	4.17	16.3	16.2	0.613
15.0	14.7	15.0	2.04	10.8	10.8	0
20.0	19.8	20.0	1.01	8.05	8.05	0
25.0	24.8	25.0	0.806	6.43	6.43	0
30.0	29.9	30.0	0.334	5.36	5.36	0
35.0	34.9	35.0	0.287	4.59	4.59	0

This table suggests that, apart from when  $x$  is small, the difference between choosing a point on the hyperbola or its asymptote is negligible. As Worsnup<sup>24</sup> suggests, placing the ball on this asymptote is much easier to find than the hyperbola, and, provided  $x$  is fairly large, the conversion angle is almost equal to the maximum.

However, there is a limitation with the investigation, as we have not considered the fact that the ball not only needs to pass through the posts but also over the crossbar.

Polster and Ross suggest that “this raises two related but distinct issues...no matter the optimal kicking location for angle width, the player does not want to kick when too close to the goals...when a try is scored directly in front of goals, players choose to kick a minimum of about 10 metres from the goals.”<sup>25</sup>

This is particularly an issue the closer to the posts because, as De Villiers states: “as a try approaches the posts, the hyperbola rapidly approaches an upright post.”<sup>26</sup> Consequently, the optimal distance and maximum angle are very close to the posts, to the point that it would be impractical to attempt to kick from such a position. However, further away, the height of the crossbar is less of an issue, as the angle of elevation required is much lower.

Consequently, one of the major drawbacks so far is that, in De Villiers’ words, it “is only likely to give reasonably good results for tries scored further from the posts.”<sup>27</sup>

However, Polster and Ross also raise concern that the angle measured so far is “simply the incorrect angle to consider...Given that the ball must clear the crossbar, it is some raised angle that should be optimised.”<sup>28</sup>

<sup>24</sup> Ibid.

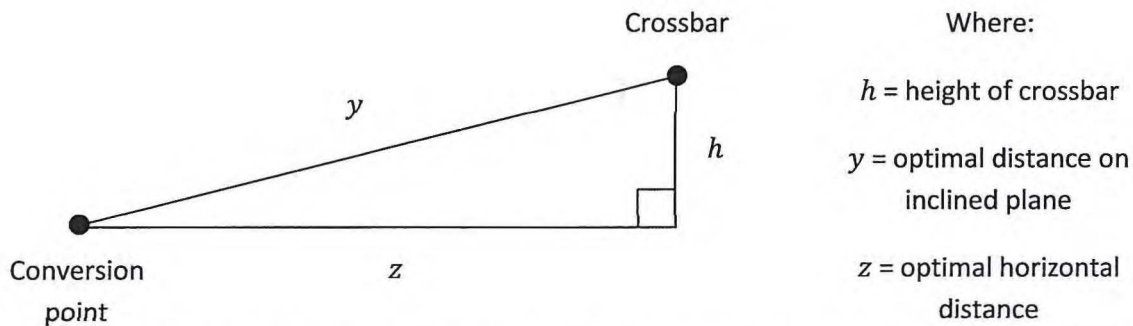
<sup>25</sup> Burkard Polster and Marty Ross (2010), Op. Cit.

<sup>26</sup> Michael de Villiers (1999), Op. Cit.

<sup>27</sup> Ibid.

<sup>28</sup> Burkard Polster and Marty Ross (2010), Op. Cit.

In light of these issues, to improve the validity of the solution, we need to consider the height of the crossbar in our model. De Villiers<sup>29</sup> suggests that the optimal points for conversions are situated where planes, on which the tangent circles lie, cut the ground. Therefore, we must consider the optimal distance we have investigated as, rather, the distance on these inclined planes from the conversion point to the crossbar, instead of the optimal horizontal distance between the two, which is what we are ultimately aiming to find. This can be demonstrated more clearly visually:



**Figure 9:** Diagram to illustrate the difference between the optimal distance on the inclined plane and the optimal horizontal distance

Since this is a right-angled triangle, Pythagoras' Theorem can be used to find the improved horizontal distance,  $z$ , between the conversion point and the crossbar:

$$y^2 = z^2 + h^2$$

Rearranging this equation to make  $z^2$  the subject:

$$z^2 = y^2 - h^2$$

Since  $y^2 = x^2 - \left(\frac{d}{2}\right)^2$ , this equation can be rewritten:

$$z^2 = x^2 - \left(\frac{d}{2}\right)^2 - h^2$$

To find the improved horizontal distance,  $z$ , we take the square root of both sides:

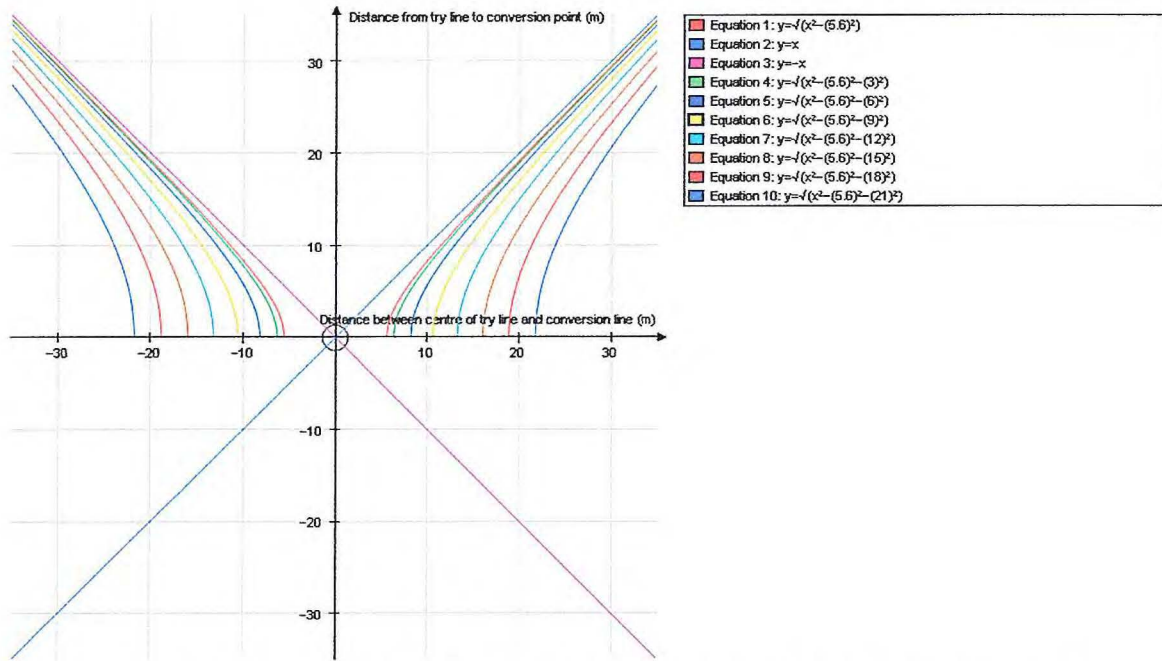
$$z = \sqrt{x^2 - \left(\frac{d}{2}\right)^2 - h^2}$$

As this is still a hyperbolic equation, this result produces an "improved hyperbolic curve on the ground"<sup>30</sup>, as De Villiers describes.

To better understand the effect of the height of the crossbar on the optimal improved horizontal distance,  $z$ , we can use technology (Autograph), inputting the graphs for various multiples of crossbar heights, with the asymptote of the hyperbolae:

<sup>29</sup> Michael de Villiers (1999), Op. Cit.

<sup>30</sup> Ibid.



**Figure 10:** Graph showing the hyperbolae for various multiples of the height of crossbar, along with the asymptotes  $y = x$  and  $y = -x$

The various graphs exemplify that, as the height of the crossbar increases, the similarity of the hyperbolae with the asymptote decreases rapidly.

As previously, we need to add values into the model to compare the results, to determine how important a factor the height of the crossbar is in finding the optimal conversion point.

Using the touchline example again, where  $x = 35m$ ,  $d = 5.6m$ , and assuming that the crossbar height is the standard  $3m$ , the value of  $z$  can be calculated:

$$z = \sqrt{x^2 - \left(\frac{d}{2}\right)^2 - h^2}$$

$$z = \sqrt{35^2 - \left(\frac{5.6}{2}\right)^2 - 3^2}$$

$$z = 34.75859606 = 34.8m \text{ (3s. f.)}$$

This value can then be substituted into the previous equation for the maximum angle at this point, only substituting  $z$  instead of  $y$ :

$$\theta_{max} = \tan^{-1}\left(\frac{(d)(z)}{z^2 + ab}\right)$$

$$\theta_{max} = \tan^{-1}\left(\frac{(5.6)(34.75859606)}{(34.75859606)^2 + (32.2 \times 37.8)}\right)$$

$$\theta_{max} = 4.588534277 = 4.59^\circ \text{ (3s. f.)}$$



These calculations suggest that, if the try is scored on the touchline, the furthest possible point from the centre of the posts, the optimal distance back from the posts at which to kick is  $34.8m$  (3s. f.), creating a maximum conversion angle of  $4.59^\circ$  (3s. f.) with the posts.

As previously, we need to create a table of values to demonstrate the results at various distances from the centre of the try line:

Distance $x$ (along try line from centre) / $m$ (3s. f.)	Optimum distance $y$ (perpendicular to try line) / $m$ (3s. f.)	Improved optimum distance $z$ (perpendicular to try line) / $m$ (3s. f.)	Maximum conversion angle $\theta$ for distance $y$ / $^\circ$ (3s. f.)	Maximum conversion angle $\theta$ for improved distance $z$ / $^\circ$ (3s. f.)
0	0	0	180	180
5.00	4.14	2.86	34.1	32.3
10.0	9.60	9.12	16.3	16.2
15.0	14.7	14.4	10.8	10.8
20.0	19.8	19.6	8.05	8.05
25.0	24.8	24.7	6.43	6.43
30.0	29.9	29.7	5.36	5.36
35.0	34.9	34.8	4.59	4.59

This table shows that the height of the crossbar can affect the optimum distance and maximum conversion angle, but only noticeably at small values of  $x$ , i.e. when the conversion line is close to the centre of the posts. When this line is further out, there is no noticeable difference is any at all, and so the height of the crossbar seems to be unimportant when the conversion has to be taken a large distance from the posts.

We have now developed two methods for optimising the distance and conversion angle, both producing identical results, and have taken into account the height of the crossbar and how it may affect the optimisation of a conversion kick. Consequently, we could say that we have achieved a solution to the problem. However, it is important not to neglect the human side to the problem. According to Polster and Ross<sup>31</sup>, "rugby kickers do not generally alter their manner of kicking in accordance with their position...they simply kick with approximately the same force and same initial angle of elevation." This suggests that the important factor to consider is a single flight path moved around the pitch depending on the location of the kicker, which could be explored in a further investigation.

<sup>31</sup> Burkard Polster and Marty Ross (2010), Op. Cit.

## Conclusion

In conclusion, I have approached the problem of optimising a conversion kick in rugby union mathematically, endeavouring to work logically and consider all factors which may affect the outcome, to achieve a correct and useful solution.

My investigation has produced several conclusions, complementing each other. Firstly, the optimum distance to place the ball back from the try line can be found in two ways:  $y_{max} = \sqrt{ab}$  and  $y = \sqrt{x^2 - \left(\frac{d}{2}\right)^2}$ , as can the maximum conversion angle at this point:  $(\theta)_{max} = \tan^{-1}\left(\frac{(d)(y)}{y^2+ab}\right)$  and  $\theta = \sin^{-1}\left(\frac{d}{2x}\right)$ . This highlights the range of methods in which this problem can be approached, and, together with the identical tables of values produced, suggests that my reasoning was correct. Furthermore, I have considered the height of the crossbar, enabling me to improve the equation for the optimal distance at which to place the ball:  $z = \sqrt{x^2 - \left(\frac{d}{2}\right)^2 - h^2}$ .

Nevertheless, these solutions would be useless if they could not be interpreted in reality, so the conclusions of this investigation are also practical. Firstly, if possible, kickers should aim to place the ball on the rectangular hyperbola  $z = \sqrt{x^2 - \left(\frac{d}{2}\right)^2 - h^2}$ , as this considers the height of the crossbar, unlike previous methods. Nonetheless, the investigation also explored how the hyperbolae have similar asymptotes, and so, if that is not possible, the ball should be placed on the asymptote.

However, its importance depends greatly on the value of  $x$ . The investigation demonstrated that, when  $x$  is small, the hyperbola moves rapidly towards the posts, and so kickers should pick the position which allows the greatest angle of elevation possible to allow the kick to be made as easily as possible, as maximum angle and range is less relevant in these areas. But, when  $x$  is large, the hyperbola is virtually asymptotic with the imaginary line at  $45^\circ$  from the posts, and so kickers should try to pick a point on this line by eye, which should be achievable.

However, the investigation has many limitations, particularly relating to its assumptions. Firstly, we have assumed the ball is a particle, whereas in reality it is an oval-shaped ball of certain mass, and so behaves differently to the assumption in the model. Secondly, we assumed that the path of the ball would be a parabola in a vertical plane, whereas kickers often swerve the ball to the left or right, depending on their kicking foot. This could greatly alter the maximum conversion angle at any point, affecting our results. Similarly, the spin on the ball was assumed to be negligible, but in reality this would almost certainly not be the case, so, depending on the direction of the spin, both the optimal distance and maximum conversion angle could be affected. Furthermore, we agreed that the kicking strength, range and accuracy are the same for the kicker at all points. However, this is never actually the case, as, the further out a kicker is, the more power they have to put into the kick at the expense of accuracy; longer kicks tend to have a smaller chance of success than those that are closer. This causes the probability to change around the pitch and so would be hard to model. Moreover, external conditions were fixed, with no weather effects caused. Realistically, this would almost never happen, so this would affect the outcome of the investigation.

Overall, the investigation has many limitations. However, without these assumptions, the results would not have been able to be found, as there would have been too many unknown variables. We had to simplify the problem to allow some form of solution to be found, albeit not perfect.

If possible, this investigation could be extended to account for some of these variables. One issue left unresolved is how to account for wind speed and direction, so we could investigate what effect wind speed and direction had on the ball in flight, and therefore on the optimal conversion position. We could also explore the biomechanical considerations mentioned earlier – kickers don't tend to change the way in which they kick depending on the conversion point, and so it might be better to model a single flight path moved around the pitch depending on the conversion point. Or we could investigate the probabilities of successful conversions around the pitch, to explore whether the optimal position can be worked out with probability rather than geometry. Moreover, we could conduct investigations in other sports, such as rugby league and American football, to identify and explore any similarities and differences in the results between sports.

Given the time and facilities to carry out, these avenues of research could reinforce and strengthen this investigation considerably.

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**Word Count:** 3,999