SUPERVISOR'S REPORT

The supervisor should complete the report below and then give this cover, enclosing the final version of the extended essay, to the Diploma Programme coordinator. The supervisor must sign this report; otherwise the extended essay will not be assessed and may be returned to the school.

Name of supervisor (CAPITAL letters)

Comments

If appropriate, please comment on the candidate's performance, the context in which the candidate undertook the research for the extended essay, any difficulties encountered and how these were overcome. These comments can help the examiner award a level for criterion H. Do not comment on any adverse personal circumstances that may have affected the candidate.

has completed a very good paper using different techniques and mathematical concepts. He began with a much focused research question and then developed his extended essay with remarkable mathematical logic. The candidate showed a high degree of initiative & enthusiasm for the topic and has done extensive research. He is focused in his approach and has dedicated a considerable amount of time and effort in writing this essay. His analysis of different methods used in scientific calculators is rather interesting and after a logical discussion, he was able to reach an insightful conclusion.

I have read the final version of the extended essay that will be submitted to the examiner.

To the best of my knowledge, the extended essay is the authentic work of the candidate.

I spent 3.5 hours with the candidate discussing the progress of the extended essay.

Supervisor's signature: _ Date:

ASSESSMENT FORM (for examiner use only)

Candidate session number								

ACHIEVEMENT LEVEL

		First examiner	maximum	Second examiner
General assessment criteria *Refer to the general guidelines.*	**A** Research question		2	
	B Approach		3	
	C Analysis/interpretation		4	
	D Argument/evaluation		4	
	E Conclusion		2	
	F Abstract		2	
	G Formal presentation		3	
	H Holistic judgement		4	
Subject assessment criteria *Refer to the subject guidelines. Not all of the following criteria will apply to all subjects; use only the criteria which apply to the subject of the extended essay.*	**J**			
	K			
	L			
	M			

TOTAL OUT OF 36

Name of first examiner (CAPITAL letters): _ Examiner number:

Name of second examiner (CAPITAL letters): ______________________ Examiner number:

The Use of Several Numerical Methods in Scientific Calculators

Abstract

All scientific calculators employ numerical methods in the execution of some operations, from the more basic to the most advanced, and yet most people are unaware of what goes on inside of the calculator to arrive at the result. It is commonly assumed that the calculator will definitely be able to arrive at the correct result. This essay seeks to establish **which numerical methods are employed in the execution of some of the calculators' most frequently used operations**, and analyse how long these methods take to work and whether they give correct results. The operations considered includes how the calculator calculates the values of trigonometric, exponential and logarithmic functions, how it solves equations, and how it does numerical integration. These operations correspond to those found on a scientific graphing calculator, the Ti-84 Plus. For each of these operations, several of the more commonly used methods are detailed and explained why they work fast and accurately. The method used in this particular calculator is also considered. All the methods are then compared against each other by applying them on a particular problem. The findings show that the Ti-84 Plus uses methods, CORDIC for the calculation of trigonometric, exponential and logarithmic functions, a combination of bisection and secant method for the solution of equations, and Gauss-Kronrod quadrature for numerical integration, that give correct results in a reasonably fast time. Therefore, it can be concluded that calculators do use methods that give accurate results, justifying peoples trust in calculators.

Contents

Appendix

Chapter 1

Introduction

More than a few decades ago, people had no calculators or computers to help them to do their calculations. They had to do their calculations on paper, often relying on ingenious tricks to simplify their calculations. Within the last century, students relied on tables of values to do their calculations on logarithms and trigonometrical functions.

Calculators nowadays are capable of executing hundreds of different functions. How can we be sure what goes on inside the calculator to come up with the result? This essay will examine **the methods that are in use in the calculations of some of the more commonly used functions** as well as examine the reliability, accuracy and speed of the results given by the calculator. The three operations that will be covered are: firstly, calculating the values of trigonometric, exponential and logarithmic function, secondly, finding the roots of equations, and lastly, doing numerical integration. The methods employed in the calculator model **Ti-84 Plus** will be compared with the other methods to understand why it was chosen. In this essay, I have chosen to focus on the **Ti-84 Plus** because I use this calculator in my Higher and Further Mathematics class.

I will explain the difference between a numerical method and an analytical method by means of an example. To find the roots of the quadratic equation $x^2 - 2 = 0$, we can factorise the equation into $\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)=0$, from which the solutions are $\pm\sqrt{2}$. These are known as the analytical answers obtained through an analytical method. On the other hand, if we had solved the problem using the CALC function in the **Ti-84 Plus**, we would have obtained either the value of $x = -1.4142136$ or $x = 1.4142136$. The fact that the value of $\sqrt{2}$ was approximated to 1.4142136 distinguishes these values as numerical answers. However, the main reason why these are numerical answers is because they were obtained using a numerical method, which deals with numerical values throughout its process, as opposed to exact values. The calculator does not use analytical methods because the calculator cannot handle the analytical answers most of the time. In addition, sometimes an answer cannot be obtained in terms of an analytical answer but is only available as a numerical answer. An example of this would be the solutions of a polynomial of degree 5 or higher. Therefore, in this essay, I will be considering numerical methods, which can be applied in all general cases.

Chapter 2

Background information on the calculator

I will start with describing the main feature of most calculators that explain why numerical methods are used. There will be a focus on the **Ti-84 Plus** and in this chapter, all reference to a calculator will apply to the **Ti-84 Plus**, although most other calculators share similar features.

Most calculators deal with so-called floating-point arithmetic. This refers to the case where results are only calculated to a certain number of significant figures, depending on the calculator. For example, on the **Ti-84 Plus**, all results are calculated to 14 significant figures. Any value less significant than the 14^{th} significant figure is essentially discarded. For example, $1 + 10^{-15} = 1.000000000000001$. However, on the **Ti-84 Plus**, $1 + 10^{-15} = 1$. This is because:

	1	$=$	1.0000000000000	|	
$+$	10^{-15}	$=$	0.0000000000000	|	01
		$=$	1.0000000000000	|	

The symbol **|** represents the border where the values to its left are within 14 significant figures and the values to its right are outside and are therefore discarded.

There are two implications of this feature. Firstly, some information is simply lost. The calculator can only and need only calculate the first 14 significant figures. Secondly, if a method involves many iterations on the same number, the error, which is the difference between the actual and calculated answer, could quickly multiply. This is the reason why most calculators display the values to less significant figures than they are capable of calculating in. The last few digits, known as "guard digits", might not contain accurate values.

For simplicity, we will not discuss the methods employed in the numerical calculations listed below, and we will simply assume these calculations to be accurate.

Where α and β are both real numbers:
$\alpha + \beta$ (addition)
$\alpha - \beta$ (subtraction)
$\alpha * \beta$ (multiplication)
$\alpha \div \beta$, $\beta \neq 0$ (division)

By extension of the above, these can be calculated too.
Where α is an integer and β is any real number:
$\beta^\alpha = \overbrace{\beta * \beta * \beta * \ldots * \beta * \beta}^{\alpha}$ (power)
$\alpha! = \alpha * (\alpha - 1) * (\alpha - 2) * (\alpha - 3) * \ldots * 2 * 1$ (factorial)

Chapter 3

Trigonometric, exponential and logarithmic functions

The functions mentioned in the title all have properties that allow the domain of the function to be reduced. The trigonometric functions are periodic with period 2π, and the exponential and logarithmic functions have properties $f(\alpha+\beta) = f(\alpha) * f(\beta)$ and $f(\alpha\beta) = f(\alpha) + f(\beta)$ respectively. Such properties allow numerical methods to essentially limit the domain on which they need to operate. For example for trigonometric functions, our methods need only produce values for inputs between 0 and $\frac{\pi}{2}$. For an exponential function, the rule $e^{2.468} = e^2 * e^{0.468}$ allows us to restrict the domain to [0,1], and for logarithmic functions, the rule $\ln(123456789) \stackrel{\sim}{=} \ln(1.8396495) + 26 * \ln(2)$ allows a reduction to values between 1 and 2.

Below, we will consider, for trigonometric and exponential functions, a number of numerical methods aimed at obtaining their values.

3.1 Trigonometry Table

A trigonometry table is a table that contains the values of a particular trigonometric function over a certain range. Before calculators were created, this was the main method people used to calculate trigonometric functions. There are also exponential tables and logarithm tables for the respective functions. An example of a trigonometry table is shown below.

m° ∠A	sin A	cos A	tan A	m° ∠A	sin A	cos A	tan A
1	0.0175	0.9998	0.0175	46	0.7193	0.6947	1.0355
2	0.0349	0.9994	0.0349	47	0.7314	0.6820	1.0724
3	0.0523	0.9986	0.0524	48	0.7431	0.6691	1.1106
4	0.0698	0.9976	0.0699	49	0.7547	0.6561	1.1504
5	0.0872	0.9962	0.0875	50	0.7660	0.6428	1.1918
6	0.1045	0.9945	0.1051	51	0.7771	0.6293	1.2349
7	0.1219	0.9925	0.1228	52	0.7880	0.6157	1.2799
8	0.1392	0.9903	0.1405	53	0.7986	0.6018	1.3270
9	0.1564	0.9877	0.1584	54	0.8090	0.5878	1.3764
10	0.1736	0.9848	0.1763	55	0.8192	0.5736	1.4281
11	0.1908	0.9816	0.1944	56	0.8290	0.5592	1.4826
12	0.2079	0.9781	0.2126	57	0.8387	0.5446	1.5399
13	0.2250	0.9744	0.2309	58	0.8480	0.5299	1.6003
14	0.2419	0.9703	0.2493	59	0.8572	0.5150	1.6643
15	0.2588	0.9659	0.2679	60	0.8660	0.50	1.7321
16	0.2756	0.9613	0.2867	61	0.8746	0.4848	1.8040
17	0.2924	0.9563	0.3057	62	0.8829	0.4695	1.8807
18	0.3090	0.9511	0.3249	63	0.8910	0.4540	1.9626
19	0.3256	0.9455	0.3443	64	0.8988	0.4384	2.0503
20	0.3420	0.9397	0.3640	65	0.9063	0.4226	2.1445
21	0.3584	0.9336	0.3839	66	0.9135	0.4067	2.2460
22	0.3746	0.9272	0.4040	67	0.9205	0.3907	2.3559
23	0.3907	0.9205	0.4245	68	0.9272	0.3746	2.4751
24	0.4067	0.9135	0.4452	69	0.9336	0.3584	2.6051
25	0.4226	0.9063	0.4663	70	0.9397	0.3420	2.7475
26	0.4384	0.8988	0.4877	71	0.9455	0.3256	2.9042
27	0.4540	0.8910	0.5095	72	0.9511	0.3090	3.0777
28	0.4695	0.8829	0.5317	73	0.9563	0.2924	3.2709
29	0.4348	0.8746	0.5543	74	0.9613	0.2756	3.4874
30	0.50	0.8660	0.5774	75	0.9659	0.2588	3.7321
31	0.5150	0.8572	0.6009	76	0.9703	0.2419	4.0108
32	0.5299	0.8480	0.6249	77	0.9744	0.2250	4.3315
33	0.5446	0.8387	0.6494	78	0.9781	0.2079	4.7046
34	0.5592	0.8290	0.6745	79	0.9816	0.1908	5.1446
35	0.5736	0.8192	0.7002	80	0.9848	0.1736	5.6713
36	0.5878	0.8090	0.7265	81	0.9877	0.1564	6.3138
37	0.6018	0.7986	0.7336	82	0.9903	0.1392	7.1154
38	0.6157	0.7880	0.7813	83	0.9925	0.1219	8.1443
39	0.6293	0.7771	0.8098	84	0.9945	0.1045	9.5144
40	0.6428	0.7660	0.8391	85	0.9962	0.0872	11.4301
41	0.6561	0.7547	0.8693	86	0.9976	0.0698	14.3007
42	0.6691	0.7431	0.9004	87	0.9986	0.0523	19.0811
43	0.6820	0.7314	0.9325	88	0.9994	0.0349	28.6363
44	0.6947	0.7193	0.9657	89	0.9998	0.0175	57.2900
45	0.7071	0.7071	1	90	1	0	Undefined

Figure 3.1: An example of a trigonometry table

Using the above table, one can find the values to 4 decimal places for any integer input. There are two disadvantages in using such a table. The first one is that there are no values for non-integer inputs and the second one is that the values are only accurate to 4 decimal places.

Both problems can be solved by including more values and storing more decimal places for each of the values. The problem with this is that the input of more data would require a larger table, and this would take up more space on the calculator's memory. This problem with increasing the table illustrates the tradeoff between the size of the table and its accuracy. For a portable calculator, the size of the memory is a very important factor in the cost of a calculator.

For the first problem, another possible solution is to interpolate the value. This allows for the finding of values for non-integer input values and is more accurate than by merely choosing the value in the table which the input value is nearer to. Linear interpolation requires only a very simple calculation and can still get accurate results. Linear interpolation can be done by using the following formula:

$$f(x_n+\Delta h) = f(x)+\Delta h*(f(x_{n+1}) - f(x_n)) = \Delta h*f(x_{n+1})+(1-\Delta h)*f(x_n)$$

For example,

$$\sin(12.3^o) = 0.3*\sin(13^o) + (1-0.3)*\sin(12^o) = 0.21303$$

This value tallies with the answer obtained by a direct calculation on the Ti-84 Plus.

The main advantage of this method is that the calculator needs only to look for the input value in the table and possibly do a simple interpolation to come up with the output. This is the fastest of all the other methods which all require more calculations. The main disadvantage is that each entry in the table takes up space in the calculator's memory. Since there are other methods that are more accurate without taking up any significant amount of time, this method is not used in a calculator, which essentially explains why calculators do not store and employ tables.

3.2 Taylor Series

Some functions can be expressed as a polynomial with an infinite number of terms. These polynomials are known as a Taylor Series. These are some of the Taylor series for the trigonometrical, exponential and logarithmic functions.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \frac{x^{17}}{17!} - \frac{x^{19}}{19!} + \ldots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} - \frac{x^{14}}{14!} + \frac{x^{16}}{16!} - \frac{x^{18}}{18!} + \ldots$$

$$\tan(x) = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + \ldots$$

x must be measured in radians.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots \text{ for } -1 < x < 1$$

As the Taylor series consist of an infinite number of terms, therefore the exact value can only be obtained when all the infinite terms have been calculated. However, the calculator cannot calculate an infinite number of terms without taking an infinite amount of time. Instead it will only calculate a certain number of the terms. The function associated with a truncated number of terms is known as the partial series of the Taylor series. For a function $f(x)$ that can be expressed by a Taylor series, the error we make by approximating $f(a)$ by the n^{th} partial series (the partial Taylor series of degree n) is $\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$, where $f^{(n+1)}(x)$ is the $(n+1)^{\text{th}}$ derivative of $f(x)$ and ξ is a value between 0 and a. Therefore, the closer a is to 0, the smaller the values ξ can take and the smaller the error.

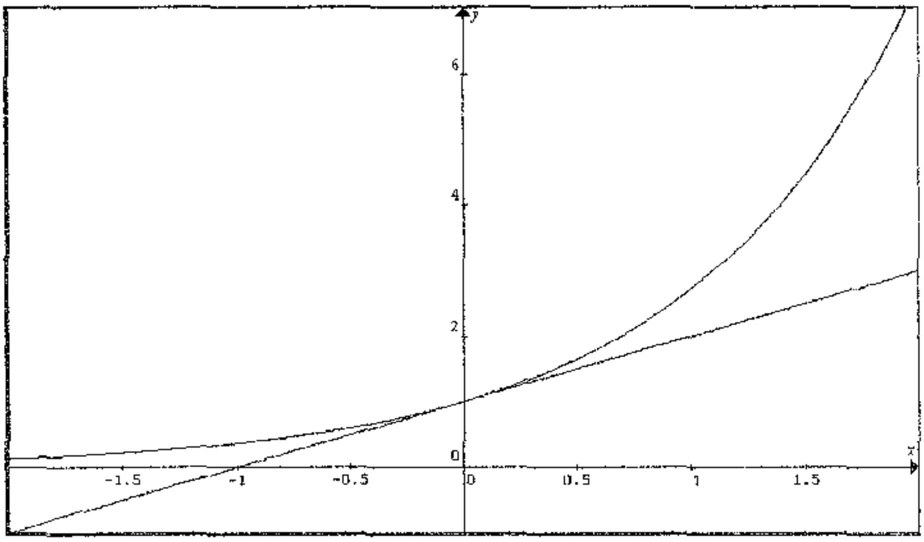

Figure 3.2: e^x and $T_2(x) = 1 + x$

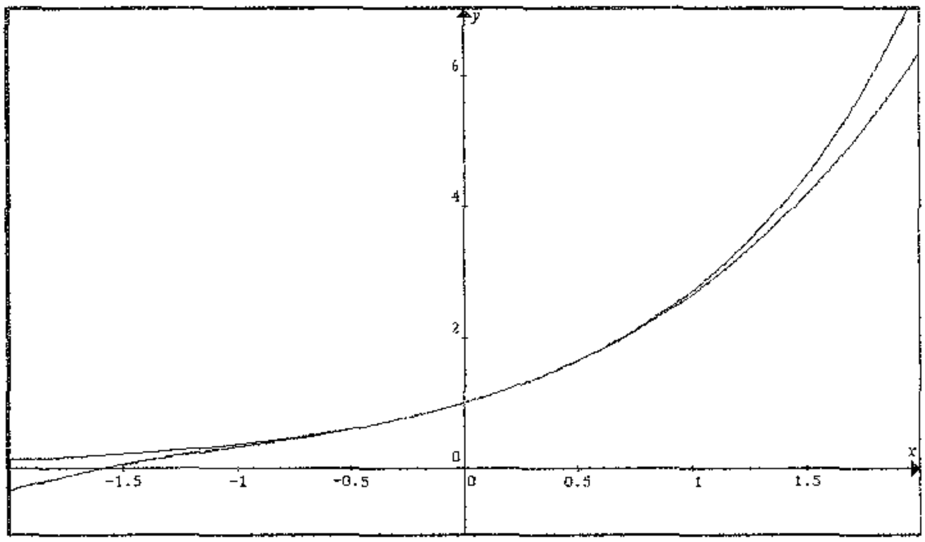

Figure 3.3: e^x and $T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

As can be seen from the above two figures, the first few partial Taylor series of e^x are a very good approximation for e^x.

For the trigonometric functions, the input can be reduced to a value between 0 and $\frac{\pi}{2}$, as explained above, where a few terms of the partial series is sufficient to obtain a very accurate answer. So the error produced by using the n^{th} Taylor series on the interval $[0, \frac{\pi}{2}]$ is maximally $\frac{1}{(n+1)!}\left(\frac{\pi}{2}\right)^{n+1}$. A quick calculation shows that for $n = 19$, the error is less than 10^{-14}. The 19^{th} Taylor series can be used to produce results accurate to 14 dp. Similarly, for e^x, the error is less than 10^{-14} for the interval $[0, 1]$ when $n = 16$ and the error for $\ln(1+x)$ is less than 10^{-14} for the interval $[0, 1]$ when $n = 45$.

The calculator needs only to store the coefficients of all the terms. This will ensure that the calculations can be done quickly.

The advantage of this method is that the calculator can calculate the values of the functions for any input value and get an accurate value. The disadvantages are that repeated calculations are needed in order to get the accuracy required and quite a lot of coefficients must be stored which will take up a fair bit of space. However, the calculator is able to do such calculations very quickly and this is why this method is very useful.

3.3 CORDIC

CORDIC stands for COordinate Rotation DIgital Computer. The concept is to take a unit vector and rotate it towards the angle required. The following explains how to obtain the sine and cosine of an angle. Slight modifications of this method allow for the calculation of other functions such as inverse trigonometric, exponential and logarithmic functions.

An anti-clockwise rotation of a vector with coordinates (X, Y) by an angle θ, in radians, results in the new coordinates (X', Y'),

$$X' = X\cos(\theta) - Y\sin(\theta) = \cos(\theta) * (X - Y\tan(\theta))$$

$$Y' = Y\cos(\theta) + X\sin(\theta) = \cos(\theta) * (Y + X\tan(\theta))$$

On the other hand, a clockwise rotation results in the new coordinates

$$X' = X\cos(\theta) + Y\sin(\theta) = \cos(\theta) * (X + Y\tan(\theta))$$

$$Y' = Y\cos(\theta) - X\sin(\theta) = \cos(\theta) * (Y - X\tan(\theta))$$

In particular, for angles of the form $\theta_k = \tan^{-1}(2^{-k})$, where $k = 0, 1, 2, \ldots, n$, we get

$$X' = \cos\left(\tan^{-1}(2^{-k})\right) * \left(X \mp Y\tan\left(\tan^{-1}(2^{-k})\right)\right) = \frac{X \mp Y * 2^{-k}}{\sqrt{1 + 2^{-2k}}}$$

$$Y' = \cos\left(\tan^{-1}(2^{-k})\right) * \left(Y \pm X\tan\left(\tan^{-1}(2^{-k})\right)\right) = \frac{Y \pm X * 2^{-k}}{\sqrt{1 + 2^{-2k}}}$$

The Taylor series of $\tan^{-1}(2^{-k})$ is:

$$\tan^{-1}(2^{-k}) = 2^{-k} - \frac{(2^{-k})^3}{3} + \frac{(2^{-k})^5}{5} - \frac{(2^{-k})^7}{7} + \frac{(2^{-k})^9}{9} - \frac{(2^{-k})^{11}}{11} + \ldots$$

For large values of k, 2^{-k} becomes very small and so its powers become small enough to be safely ignored. Therefore, for large values of k, $\theta_k = \tan^{-1}(2^{-k}) \approx 2^{-k}$. This allows the calculator to save memory needed from storing all the values of θ_k.

Starting from the zero angle, the unit vector is rotated towards the input angle, call it α, through all the values of θ_k. The unit vector is rotated anti-clockwise by θ_k, unless the previous rotation rotated the unit vector pass the input angle, in which case the unit vector is instead rotated clockwise by θ_k.

$$X_0 = 1, Y_0 = 0, \alpha_0 = \alpha$$

If $\alpha_k > 0$

$$X_{k+1} = \frac{X_k - Y_k * 2^{-k}}{\sqrt{1 + 2^{-2k}}}$$

$$Y_{k+1} = \frac{Y_k + X_k * 2^{-k}}{\sqrt{1 + 2^{-2k}}}$$

$$\alpha_{k+1} = \alpha_k - \theta_k$$

If $\alpha_k < 0$

$$X_{k+1} = \frac{X_k + Y_k * 2^{-k}}{\sqrt{1 + 2^{-2k}}}$$

$$Y_{k-1} = \frac{Y_k - X_k * 2^{-k}}{\sqrt{1 + 2^{-2k}}}$$

$$\alpha_{k+1} = \alpha_k + \theta_k$$

where α_k keeps track of whether to rotate clockwise or anti-clockwise.

By the definition of sine and cosine, $X = \cos(\alpha)$ and $Y = \sin(\alpha)$. At large values of k, the difference of (X_k, Y_k) from (X, Y) is at most $2^{-(k+1)}$, so the same is true for the error we make in $\cos(\alpha)$ and $\sin(\alpha)$, so to achieve accuracy till 14 dp, all we need to do is make sure that k is so large that $2^{-(k+1)} < 10^{-14}$, i.e $k = 46$. The method above needs only about k steps to reach the required accuracy.

This method cannot be used on a large value of α, since the sum of all the angles of $\tan^{-1}(2^{-k})$ does not exceed 1.744, or slightly more than $\frac{\pi}{2} = 1.571$. Therefore, this method only works on angles between 0 and $\frac{\pi}{2}$, but as we remarked early in this section, that is not a problem, as the basic trigonometric properties allow the reduction of any angle to one in this interval.

The **Ti-84 Plus** uses the CORDIC method with some differences, which will be explained in more details in the appendix. The important thing to note is that the modified method can be done more quickly than the original method.

The appendix contains a comparison of how the various methods fare in calculating the sine and cosine of $\frac{\pi}{8}$.

Chapter 4

Solutions of the equations

Solving an equation $f(x) = g(x)$ is to find the values of x that will give the same values for $f(x)$ and $g(x)$. Geometrically, this is equivalent to finding the intersection of the curves when the two functions are plotted on a graph.

When $g(x)$ is 0, then the equation becomes $f(x) = 0$. The values of x where $f(x) = 0$ are known as the roots of the function $f(x)$. When $g(x)$ is another function, one can subtract $g(x)$ on both sides such that the equation becomes $h(x) = f(x) - g(x) = 0$. The solutions of the equation are the roots of $h(x)$. Therefore, finding the solutions of an equation is the same as finding the roots of the function $f(x) - g(x)$.

The three methods described below do repeated iterations where they approximate the root closer and closer at each iteration.

4.1 Bisection Method

Assuming that the function $f(x)$ is continuous, i.e. there are no sudden breaks, then if there are two points α and β such that $f(\alpha)$ is positive and $f(\beta)$ is negative or vice-versa (they have different signs), then there exists a root between the interval of α and β. A root can exist between two points where the values of the function have the same sign. An example of this would be $f(x) = x^2$. In such a situation, this method will not be able to find the root.

To find the root, the calculator can take the midpoint of α and β, call it $\gamma = \frac{\alpha+\beta}{2}$, and see whether $f(\gamma)$ has the same sign as $f(\alpha)$ or $f(\beta)$. If $f(\gamma)$ has the same sign as $f(\alpha)$, then a root must be between $f(\beta)$ and $f(\gamma)$. Similarly, if $f(\gamma)$ has the same sign as $f(\beta)$, then a root must be between $f(\alpha)$ and $f(\gamma)$. Of course, if $f(\gamma) = 0$, then the root has been found and the process can be stopped. Thus an interval for the root that is half the original size can be found and we can repeat this step again and again until the interval converges to the root of the function.

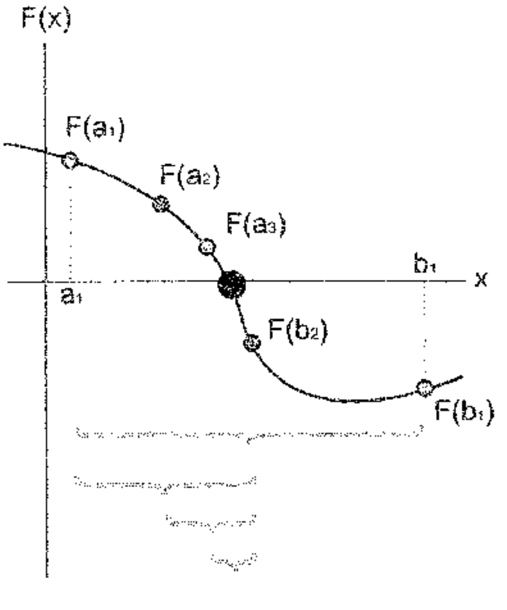

Figure 4.1: Graphical display of the bisection method

The calculator can implement this method by asking for the user to input a left bound and right bound between which a root can be found. The calculator can then repeatedly half the interval until the left and right bound are the same values at the accuracy of the calculator. With an accuracy of 14 sf, about 50 iterations are sufficient to converge on the root.

The advantage of this method is that it will definitely find a root of the function if there exists a sign change to the left and the right of the root. This method is fast although compared to the other methods, it is the slowest.

4.2 Newton-Rhapson

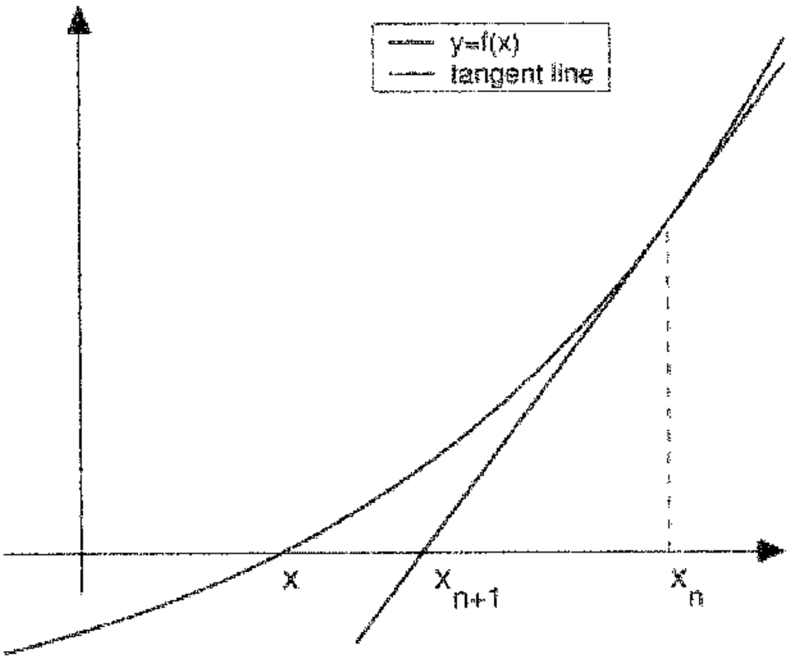

Figure 4.2: Graphical display of the Newton-Rhapson method

Given a function $f(x)$, at the point x_0, near the actual root, a better approximation of the root might be obtained by taking the point where the tangent of the curve at x_0 cuts the x-axis, at x_1. Then x_1 takes the role of x_0 and we repeat this procedure again and again and the points should get closer and closer to the root.

The gradient of the tangent line is given by $\frac{f(x_0) - 0}{x_0 - x_1}$. The gradient is also given by the derivative of the function which is $f'(x_0)$.

$$\Rightarrow f'(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$x_0 - x_1 = \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Therefore, the generation of the points would follow the recursive formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The first approximation can be obtained by asking the user to input a value close to the actual root or have the calculator run through values of x. The calculator can then repeatedly iterate until the answer reaches the accuracy required, meaning that the difference between x_{n+1} and x_n is less than 10^{-15} or so.

This method finds the root very quickly if the first approximation is near the root and the gradient is not near zero. If the gradient is zero, the method will fail. If the gradient is near zero, the next point obtained will deviate largely from the actual root. If the user does not choose the first approximation properly, the values of x_n might not be able to reach the root or end up finding a root other than the root that the user wanted.

As a result of the disadvantages, this method is not used in the **Ti-84 Plus**.

4.3 Secant method

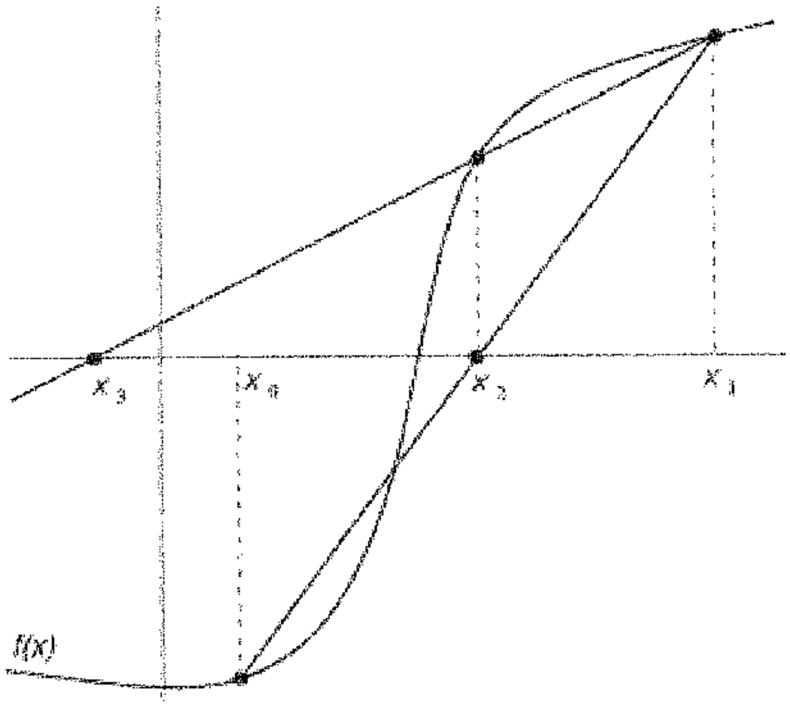

Figure 4.3: Graphical display of the Secant method

Given a function $f(x)$, at two points x_0 and x_1, one to the left and one to the right of the actual root, a better approximation of the root might be obtained by taking the point where the line that passes through $(x_0, f(x_0))$ and $(x_1, f(x_1))$, known as the secant line, cuts the x-axis, at x_2. Sub x_1 into x_0 and x_2 into x_1 and repeat. Thus, the points get closer and closer to the actual root.

The equation of the secant line is

$$y - y_1 = m * (x - x_1)$$

$$y - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} * (x - x_1)$$

At $(x_2, 0)$,

$$0 - f(x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} * (x_2 - x_1)$$

$$x_2 - x_1 = \frac{-f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{-f(x_1) * (x_1 - x_0)}{f(x_1) - f(x_0)} + x_1$$

$$x_2 = \frac{-x_1 * f(x_1) + x_o * f(x_1) + x_1 * f(x_1) - x_1 * f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{x_0 * f(x_1) - x_1 * f(x_0)}{f(x_1) - f(x_0)}$$

Therefore, the generation of the points would follow the recursive formula:

$$x_{n+1} = \frac{x_{n-1} * f(x_n) - x_n * f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

The advantage of this method is that it is faster than the bisection method to converge towards the root if it does converge. However, like the Newton-Raphson method, it suffers from some of the same problems. If the first two approximations are not near the root, the method might not find the root the user is looking for. The secant line might cut the x-axis at a point far away from the two points used, especially if the gradient is close to zero and the first two points chosen have function values of the same sign.

As a result of the disadvantages, this method is also not used in the **Ti-84 Plus.**

The **Ti-84 Plus** uses a combination of the bisection method and the secant method, as will be further detailed in the appendix. The appendix also contains a comparison of how the various methods fare in calculating the solution of $\sin(x) = \frac{1}{\sqrt{2}}$.

Chapter 5

Numerical integration

Integrating a function between the points α and β, also known as the definite integration, is the calculating of the area of the function between α and β, bounded by the x-axis and the curve. Along with the function itself, the values of α and β are given by the user. Integration is another example of a case where there is no analytical answer for some functions. For example, there is no exact answer for $\int e^{-x^2} dx$. Numerical integration can give the answer of this integration to any number of decimal places as required or as limited by the calculator.

All the methods take two approximations of the area of the curve. One is the area through the interval required. The other is the sum of the areas of two halves of the interval. If the difference in the area is larger than the accuracy required, the interval will be halved and the approximation done over each interval. The reason for reducing the intervals is that the approximation is closer to the actual result for smaller intervals. The intervals will be referred to as $[\alpha = x_0, x_1, x_2, x_3, \ldots, x_n = \beta]$, the length of each interval as $h = x_n - x_{n-1}$.

5.1 Trapezoid rule

We can approximate a curve by a straight line cutting the left and right ends of an interval. The area under this straight line is the area of a trapezium, hence the name of the method. The area would be

$$h*\left(\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\frac{f(x_2)+f(x_3)}{2}+\ldots+\frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$=\frac{h}{2}*(f(x_0)+2(f(x_1)+f(x_2)+\ldots+f(x_{n-1}))+f(x_n))$$

This method will give an exact answer if the function is linear between intervals. This method would thus give an exact answer on a linear function. Like the previous method, this method is not very accurate for most functions unless there is a high enough number of intervals. Since there are other better methods, this method is not used.

The diagram below shows how the method looks graphically.

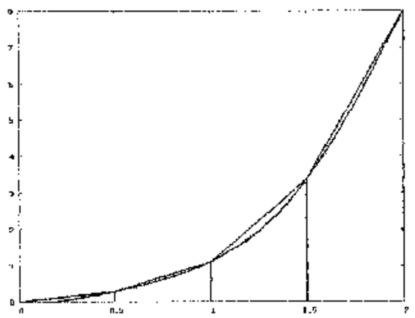

Figure 5.1: Trapezoidal method on the function $f(x) = x^3$ with 4 intervals between [0,2]

5.2 Simpson's rule

Instead of approximating each interval by a straight line, if the interval is approximated by a quadratic polynomial, then the approximation will be more accurate. The area over the whole interval would be

$$\frac{h}{3}\Big(f(x_0)+f(x_n)+4\Big(f(x_1)+f(x_3)+\ldots+f(x_{n-1})\Big)$$
$$+2\Big(f(x_2)+f(x_4)+\ldots+f(x_{n-2})\Big)\Big)$$

As the above formula shows, n must be an even number. Therefore, the number of intervals must be even for this method to work. This method will give an exact answer if the function is quadratic between intervals. This method would thus give an exact answer on a quadratic function.

The Simpson's rule is an easy method to use and gives remarkably accurate results after subdividing the interval for a few times. Some calculators use this method. There is another method that takes less time to give a more accurate result and this other method is used in the **Ti-84 Plus** instead of the Simpson's rule.

5.3 Gauss-Kronrod quadrature

This is the method that the **Ti-84 Plus** use to calculate integration. Unlike the previous methods, the Gauss-Kronrod method does not take the values of the function at equally spaced intervals. Instead, it takes the weighted sum of the function calculated at certain points. Weighing a value means to multiply the value by another number to take into account difference in significance.

The larger the number of points that the method takes into account, the more accurate the result. Taking n points can give exact results of the integration of a polynomial of $2n-1$ degree.

The interval for the integration is normally given as [-1,1].

$$\int_{-1}^{1} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + w_4 f(x_4) + \ldots + w_n f(x_n)$$

As an example, a three point Gauss-Kronrod rule would use the point x = 0 with weight $\frac{8}{9}$ and the points $x = \pm\sqrt{\frac{3}{5}}$ with weight $\frac{5}{9}$.

However, since most integrations do not take the interval of [-1,1], there needs to be a way to change the interval from [-1,1] to $[\alpha, \beta]$.

The formula is

$$\int_{\alpha}^{\beta} f(x)dx = \frac{\beta-\alpha}{2} * \int_{-1}^{1} f\left(\frac{\beta-\alpha}{2}x + \frac{\alpha+\beta}{2}\right) dx$$

$$\approx \frac{\beta-\alpha}{2} * \sum_{i=0}^{n} \left(w_i f\left(\frac{\beta-\alpha}{2}x_i + \frac{\alpha+\beta}{2}\right)\right)$$

$$= u * \left(w_0 f\left(ux_0 + v\right) + w_1 f\left(ux_1 + v\right) + \ldots + w_n f\left(ux_n + v\right)\right)$$

$$\text{where } u = \frac{\beta-\alpha}{2} \text{ and } v = \frac{\alpha+\beta}{2}$$

Each time that the interval is reduced, there is a need to recalculate the values of every single point, unlike the previous methods. This would slow down the process very much. However, research was done by Alexander Kronrod to use points such that most of the points before division of the intervals coincide with the points after the division. This allows the points to be reused so there is no need to repeatedly calculate many different points.

The **Ti-84 Plus** calculator essentially uses this method to calculate the numerical integration of any function. It uses a pair of Gauss-Kronrod rules of different number of points, the larger one is the one that will provide the answer, and the smaller one is used as an approximation of the error.

Chapter 6

Conclusion

The first operation, calculating the values of trigonometric, exponential and logarithmic functions, showed two calculation methods. That is the Taylor series and the CORDIC method. Both of them are feasible methods. Both methods are used in calculators as they can calculate the values quickly.

The second operation, finding the roots of equations, showed three methods. That is the Bisection method, the Newton-Rhapson method and the Secant method. The weakness of all these three methods is that it fails when there are repeated roots. In the case of roots of odd multiplicity, the bisection method will definitely obtain the answer, and the other two methods will also obtain the answer given reasonably close enough initial estimates. Considering that the bisection method will obtain the answer and is still fast enough, it is understandable why the **Ti-84 Plus** uses the bisection method and secant method over the faster but less trustable Newton-Rhapson method.

The third operation, finding the integration of a function, showed three methods. That is the Trapezoid rule, the Simpson's rule and the Gauss-Kronrod quadrature. The first two methods are methods that can be easily done on a standard scientific calculator. The Simpson's rule is a method that will approximately obtain the actual answer with a few calculations. The Gauss-Kronrod quadrature, on the other hand, is not an easy method to apply on a calculator. Given its faster speed, if it can be programmed onto the calculator or come preprogrammed, such as that on the **Ti-84 Plus**, then it is a better method for doing numerical integration.

The essay has shown that the methods used in the **Ti-84 Plus** are accurate although they might not be the fastest method or work all the time.

It is hoped that from this essay, people will have gained an understanding in how the calculator is capable of doing its operations.

Chapter 7

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Appendix A

The modified CORDIC method used in the Ti-84 Plus

The method used by the **Ti-84 Plus** will be called the modified method. The original method uses angles $\theta_k = \tan^{-1}(2^{-k})$, while the modified method uses angles $\tan^{-1}(10^{-k})$. In the original method, the rotation can be either in the clockwise or anti-clockwise direction for each k, but in the modified method, we rotate in the anti-clockwise direction up to 10 times for each k.

The original CORDIC method relied on the fact that computers operate in binary mode. In the binary system that the CORDIC method was created for, each iteration only requires max two angle updates digit shifts and two (X, Y) updates, this makes the method very speedy. On the **Ti-84 Plus**, which uses the decimal system, the angles have to be changed such that the calculator is still able to maintain high speeds, therefore the angles have to be of the form $\tan^{-1}(10^{-k})$ instead.

As a result of using smaller angles, $\tan^{-1}(10^{-k})$ over $\tan^{-1}(2^{-k})$, it is impossible to achieve some angles. The sum of every angle is only .896 at most, not even 60% of $\frac{\pi}{2}$. To achieve all possible angles, some angles must be repeated. In fact, the choice of whether to rotate by an angle makes it superior to the original method. In the original method, rotating by θ_k when the difference between the desired angle and the cumulative angle is small would cause the resultant difference to be so large that the next few smaller angles have to be used to reduce the difference. The modified method overcomes this problem by only allowing the vector to be rotated in one direction, skipping angles if they are larger than the difference.

In addition, the modified method skips the need for division by $\sqrt{1+2^{-2k}}$ at each step. Without the division, (X_k, Y_k) will not be the coordinates of the unit vector but rather the coordinate of a vector that is $1+2^{-2k}$ as large as the length of the vector defined by (X_{k-1}, Y_{k-1}). By the time the method ends, the final vector will not be a unit vector. However, since the magnitude is the same in both the X and Y coordinates, one can take Y divided by X to get the value of $\tan(\alpha)$. $\sin(\alpha)$ and $\cos(\alpha)$ can then be calculated by using the formula $\sin(\alpha) = \frac{\tan(\alpha)}{\sqrt{1+\tan^2(\alpha)}}$ and $\cos(\alpha) = \frac{1}{\sqrt{1+\tan^2(\alpha)}}$. By calculating $\tan(\alpha)$ first, this step cleverly reduces inaccuracies that occur with the repeated division of $\sqrt{1+2^{-2k}}$ at each step of the method, with the additional benefit of less time wasted per step.

Appendix B

Comparison of the methods for trigonometric, exponential and logarithmic functions

Let us take a look at how the three methods described above compare to find the sine and cosine of $\frac{\pi}{8}$ as compared to an analytical answer.

$$\sin(\frac{\pi}{8}) = \sin(\frac{\frac{\pi}{4}}{2}) = \sqrt{\frac{1-\cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\Rightarrow \sin(\frac{\pi}{8}) = 0.382,683,432,365,090\ldots$$

$$\cos(\frac{\pi}{8}) = \cos(\frac{\frac{\pi}{4}}{2}) = \sqrt{\frac{1+\cos(\frac{\pi}{4})}{2}} = \sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\Rightarrow \cos(\frac{\pi}{8}) = 0.923,879,532,511,287\ldots$$

Linear interpolation using the Trigonometry Table

$$\sin(\frac{\pi}{8}) = \sin(22.5^o) = \sin(22^o) + 0.5 * (\sin(23^o) - \sin(22^o))$$

$$\sin(\frac{\pi}{8}) = 0.3746 + 0.5 * (0.3907 - 0.3746) = \mathbf{0.382{,}65}$$

$$\cos(\frac{\pi}{8}) = \cos(22.5^o) = \cos(22^o) + 0.5 * (\cos(23^o) - \cos(22^o))$$

$$\cos(\frac{\pi}{8}) = 0.9272 + 0.5 * (0.9205 - 0.9272) = \mathbf{0.923{,}85}$$

The calculated answers are accurate to 4 significant figures, which is only as accurate as the accuracy of the values given in the table.

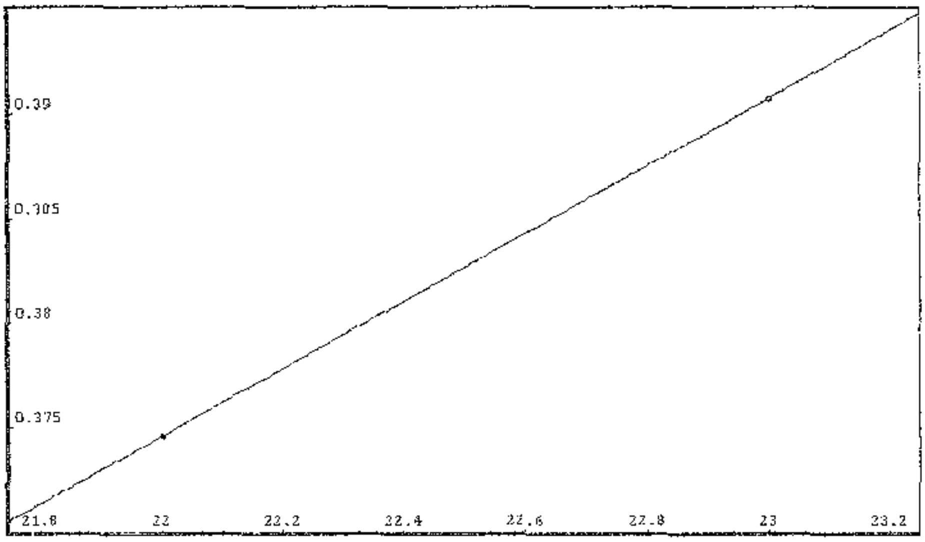

Figure B.1: Graphical display of interpolation of sin(22.5°)

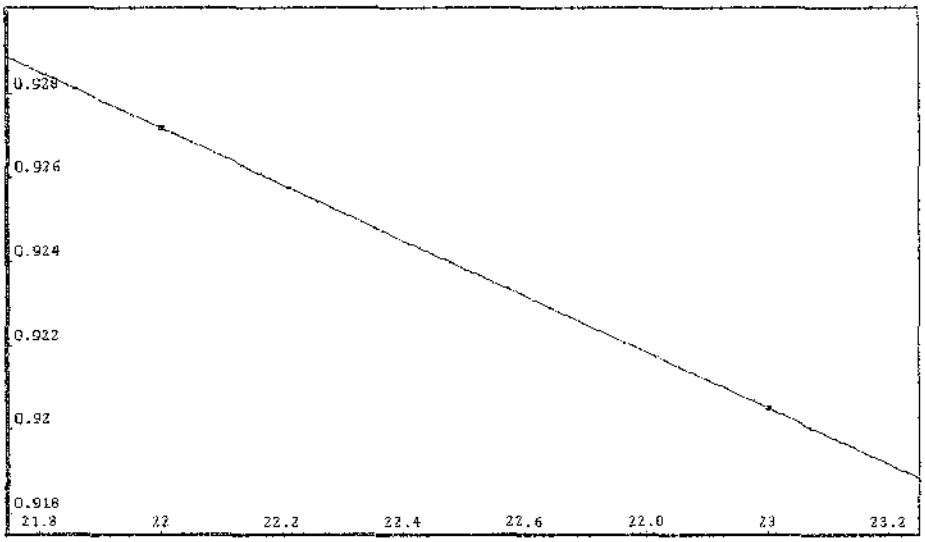

Figure B.2: Graphical display of interpolation of cos(22.5°)

Taylor expansion for sine

$$
\begin{aligned}
x &= \frac{\pi}{8} \\
&= 0.\mathbf{392},699,081,698,724 \\
x - \frac{x^3}{3!} &= \frac{\pi}{8} - \frac{\pi^3}{8^3 * 3!} \\
&= 0.\mathbf{382,6}05,892,675,189 \\
x - \frac{x^3}{3!} + \frac{x^5}{5!} &= \frac{\pi}{8} - \frac{\pi^3}{8^3 * 3!} + \frac{\pi^5}{8^5 * 5!} \\
&= 0.\mathbf{382,683},717,505,508 \\
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} &= \frac{\pi}{8} - \frac{\pi^3}{8^3 * 3!} + \frac{\pi^5}{8^5 * 5!} - \frac{\pi^7}{8^7 * 7!} \\
&= 0.\mathbf{382,683,43}1,753,912 \\
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} &= 0.\mathbf{382,683,432,365},947 \\
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} &= 0.\mathbf{382,683,432,365,089} \\
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} &= 0.\mathbf{382,683,432,365,090}
\end{aligned}
$$

Taylor expansion for cosine

$$
\begin{aligned}
1 &= 1 \\
1 - \frac{x^2}{2!} &= 1 - \frac{\pi^2}{8^3 * 2!} \\
&= 0.\mathbf{9}92,893,715,616,489 \\
1 - \frac{x^2}{2!} + \frac{x^4}{4!} &= 1 - \frac{\pi^2}{8^2 * 2!} + \frac{\pi^4}{8^4 * 4!} \\
&= 0.\mathbf{923},884,612,131,728 \\
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} &= 1 - \frac{\pi^2}{8^2 * 2!} + \frac{\pi^4}{8^4 * 4!} - \frac{\pi^6}{8^6 * 6!} \\
&= 0.\mathbf{923,879,5}18,508,495 \\
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} &= 0.\mathbf{923,879,532},535,293 \\
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} &= 0.\mathbf{923,879,532,511,2}59 \\
1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} &= 0.\mathbf{923,879,532,511,287}
\end{aligned}
$$

As the above examples show, using the Taylor series to calculate trigonometric functions (as well as the other functions) allows one to gain accurate answers very quickly.

CORDIC for both cosine and sine

k	Cumulative angle	X	Y	Rotate
0	0	1	0	Clockwise
1	0.785398163397	0.707106781187	0.707106781187	Anticlockwise
2	0.321750554397	0.948683298051	0.316227766017	Clockwise
3	0.566729217524	0.843661487732	0.536875492193	Anticlockwise
4	0.442374222977	0.903737838894	0.428086344739	Anticlockwise
5	0.379955412981	0.928681172892	0.370879062656	Clockwise
6	0.411195246411	0.916643731456	0.399705228366	Anticlockwise
7	0.395571517791	0.922776488909	0.385335634891	Anticlockwise
8	0.387759176730	0.925758672075	0.378114904597	Clockwise
9	0.391665406862	0.924274609100	0.381728237065	Clockwise
10	0.393618529379	0.923527284650	0.383532729381	Anticlockwise
11	0.392641967189	0.923901387781	0.382630664815	Clockwise
12	0.393130248401	0.923714446286	0.383081742872	Anticlockwise
13	0.392886107780	0.923807944571	0.382856215240	Anticlockwise
14	0.392764037469	0.923854673065	0.382743442864	Anticlockwise
15	0.392703002312	0.923878032150	0.382687054537	Anticlockwise
16	0.392672484734	0.923889710402	0.382658859839	Clockwise
17	0.392687743523	0.923883871384	0.382672957232	Clockwise
18	0.392695372918	0.923880951794	0.382680005896	Clockwise
19	0.392699187615	0.923879491979	0.382683530219	Anticlockwise
20	0.392697280266	0.923880221888	0.382681768058	Clockwise
21	0.392698233941	0.923879856934	0.382682649139	Clockwise
22	0.392698710778	0.923879674456	0.382683089679	Clockwise
23	0.392698949197	0.923879583218	0.382683309949	Clockwise
24	0.392699068406	0.923879537598	0.382683420084	Clockwise
25	0.392699128010	0.923879514789	0.382683475152	Anticlockwise
26	0.392699098208	0.923879526193	0.382683447618	Anticlockwise
27	0.392699083307	0.923879531896	0.382683433851	Anticlockwise
28	0.392699075856	0.923879534747	0.382683426967	Clockwise
29	0.392699079582	0.923879533321	0.382683430409	Clockwise
30	0.392699081444	0.923879532609	0.382683432130	Clockwise
31	0.392699082376	0.923879532252	0.382683432990	Anticlockwise
32	0.392699081910	0.923879532430	0.382683432560	Anticlockwise
33	0.392699081677	0.923879532520	0.382683432345	Clockwise
34	0.392699081794	0.923879532475	0.382683432453	Anticlockwise
35	0.392699081735	0.923879532497	0.382683432399	Anticlockwise

36	0.392699081706	0.923879532508	0.382683432372	Anticlockwise
37	0.392699081692	0.923879532514	0.382683432359	Clockwise
38	0.392699081699	0.923879532511	0.382683432365	Anticlockwise
39	0.392699081695	0.923879532513	0.382683432362	Clockwise
40	0.392699081697	0.923879532512	0.382683432364	Clockwise
41	0.392699081698	0.923879532512	0.382683432364	Clockwise
42	0.392699081699	0.923879532511	0.382683432365	Clockwise
43	0.392699081699	0.923879532511	0.382683432365	Anticlockwise
44	0.392699081699	0.923879532511	0.382683432365	Clockwise

Even though it takes more steps for the CORDIC method to obtain the same answer as compared to the Taylor series, it actually takes less time for each step since multiplication of powers of 10 just involve shifting the numbers to the left/right. The difference is not noticeable with just one calculation, but when graphing trigonometric equations, the CORDIC method would be faster. In addition, the values of the values of θ_k take up less space than to store the values of the coefficients of the Taylor series, as mentioned earlier.

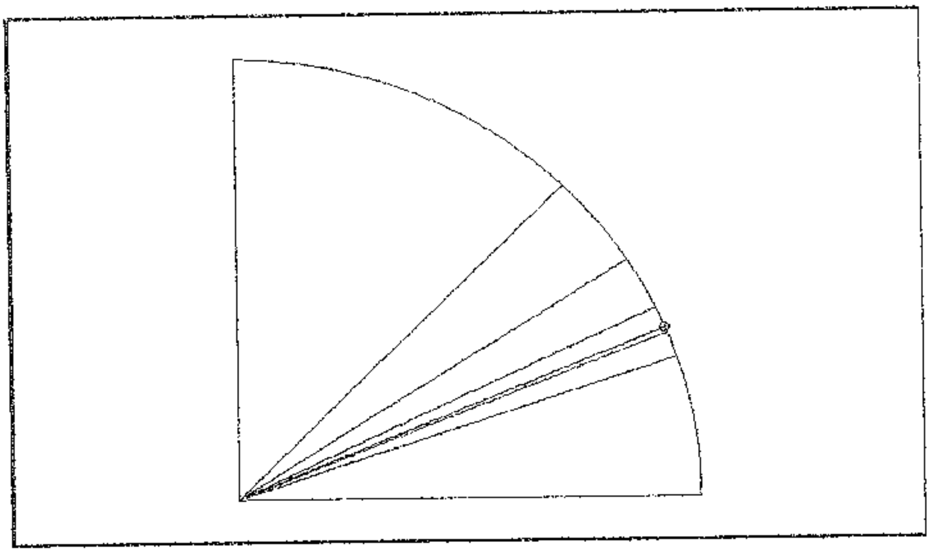

Figure B.3: Graphical display of the first five iterations of the CORDIC method

Appendix C

Method used in the Ti-84 Plus for the solutions of the equations

The **Ti-84 Plus** uses a modification of the secant method also known as the false position method together with the bisection method to find a root of the equation. An interval of a root can be obtained either by input from the user or from allowing the calculator to search for values of x with a difference in sign. From the two points, the secant method is used to obtain a third x-value. Like the bisection method, the next two points are chosen from the three points such that they form an interval between which a root can be found. The bisection method is also used on the original two points to obtain another x-value. The calculator will choose the smaller interval of the two methods. This process is repeated again and again until the difference between the left and right bound is smaller than the accuracy required.

This method has both the advantages of the two methods involved. Firstly, it will definitely find a root if there exists a sign change to the left and the right of the root. Secondly, it is slightly faster since it uses the secant method which is faster than the bisection method.

Unfortunately, this method would fail against the simplest of solutions such as $x^2 = 0$. The only way to solve the situation would be to draw the graph of the equation, and then continually zoom in at the root till the accuracy required. This would actually be the bisection method done manually.

Appendix D

Comparison of the methods for the solutions of the equations

Let us take a look at how the methods described above compare to find the solutions of $\sin(x) = \frac{1}{\sqrt{2}}$, $[0, \frac{\pi}{2}]$.

$$\Rightarrow x = \sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$$

$$\Rightarrow x = 0.785, 398, 163, 397, 448, 309, 62$$

Bisection method

$$\text{Let } f(x) = \sin(x) - \frac{1}{\sqrt{2}} = 0$$

Try values of x to find sign change.

$$f(0) = \sin(0) - \frac{1}{\sqrt{2}} < 0$$

$$f(1) = \sin(1) - \frac{1}{\sqrt{2}} > 0$$

A sign change exists between $x = 0$ and $x = 1$ $\Rightarrow$a root lies between 0 and 1. $\Rightarrow$[0,1]

1.$f(0.5) = \sin(0.5) - \frac{1}{\sqrt{2}} = -0.227, 681, 242, 582, 344, 524, 13 \Rightarrow [0.5, 1]$

2.$f(0.75) = \sin(0.75) - \frac{1}{\sqrt{2}} = -0.025, 468, 021, 163, 213, 357, 67 \Rightarrow [0.75, 1]$

3.$f(0.875) = \sin(0.875) - \frac{1}{\sqrt{2}} = 0.060, 436, 721, 049, 479, 515, 23 \Rightarrow [0.75, 0.875]$

4.$f(0.8125) = 0.\mathbf{018}, 901, 874, 074, 165, 025, 26 \Rightarrow [0.75, 0.8125]$
5.$f(0.78125) = -0.\mathbf{002}, 939, 269, 732, 013, 851, 62 \Rightarrow [0.78125, 0.8125]$
6.$f(0.796875) = 0.\mathbf{008}, 068, 602, 077, 460, 107, 86 \Rightarrow [0.78125, 0.796875]$
7.$f(0.7890625) = 0.\mathbf{002}, 586, 324, 177, 352, 200, 56 \Rightarrow [0.78125, 0.7890625]$
8.$f(0.78515625) = -0.\mathbf{000}, 171, 079, 292, 811, 027, 53 \Rightarrow [0.78515625, 0.7890625]$
Although this method will converge, but it converges slowly. The number of accurate figures increase by one every three steps on average. One thing to notice is that the values do not necessarily get more accurate at each step (look at the fifth and sixth step).

Newton method

$$f'(x) = \cos(x)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_0 = 0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{\sin(0) - \frac{1}{\sqrt{2}}}{\cos(0)} = 0.\mathbf{7}07, 106, 781, 186, 547, 524, 40$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.\mathbf{78}2, 700, 665, 480, 562, 314, 18$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.\mathbf{785{,}39}4, 541, 456, 322, 162, 24$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.\mathbf{785{,}398{,}163{,}39}0, 889, 120, 45$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.\mathbf{785{,}398{,}163{,}397{,}448{,}309{,}62}$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 0.\mathbf{785{,}398{,}163{,}397{,}448{,}309{,}62}$$

As can be seen from this example, the number of accurate figures roughly double at every iteration. If this method conerges at all, it converges rapidly. However, let us try the starting value of 1.5.

$$x_0 = 1.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -2.605, 169, 536, 980, 594, 698, 59$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -4.022, 402, 540, 074, 553, 274, 46$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -3.921,624,579,720,563,099,38$$

As can be seen, the starting value of 1.5 causes the method to converge to another root instead.

Secant method

$$x_0 = 0$$

$$x_1 = 1$$

$$x_2 = \frac{x_0 * f(x_1) - x_1 * f(x_0)}{f(x_1) - f(x_0)} = 0.840,322,238,024,614$$

$$x_3 = \frac{x_1 * f(x_2) - x_2 * f(x_1)}{f(x_2) - f(x_1)} = 0.777,928,281,477,152$$

$$x_4 = \mathbf{0.785},611,659,518,236$$

$$x_5 = 0.\mathbf{785,398},955,992,012$$

$$x_6 = \mathbf{0.785,398,163,3}10,820$$

$$x_7 = 0.\mathbf{785,398,163,397,448}$$

As can be seen from the example, the number of accurate figures roughly increase by 3 at each iteration. The rate of convergence for the secant method is slower than that of the Newton-Rhapson method but faster than the Bisection method.

Appendix E

Comparison of the methods for numerical integration

Let us take a look at how all the methods described above compare to find the definite integration of $f(x) = \cos(x)$ between -1.5 and 1.5.

$$\int_{-1.5}^{1.5} \cos(x)dx = [\sin(x)]_{-1.5}^{1.5} = \sin(1.5) - \sin(-1.5) = 2\sin(1.5)$$

$$\Rightarrow \int_{-1.5}^{1.5} \cos(x)dx = 1.994, 989, 973, 208, 108, 861, 88\ldots$$

Trapezoid rule

n	h	Area
1	3	$0.212, 211, 605, 003, 108, 730, 26$
2	1.5	$1.606, 105, 802, 501, 554, 365, 13$
4	0.75	$\mathbf{1.900}, 586, 204, 561, 508, 512, 03$
8	0.375	$\mathbf{1.971}, 556, 206, 313, 989, 606, 79$
16	0.1875	$\mathbf{1.989}, 141, 848, 524, 599, 459, 14$

Simpson's rule

n	h	Area
2	1.5	$2.070, 737, 201, 667, 702, 910, 09$
4	0.75	$\mathbf{1.998}, 746, 338, 581, 493, 227, 67$
8	0.375	$\mathbf{1.995}, 212, 873, 564, 816, 638, 38$
16	0.1875	$\mathbf{1.995}, 003, 729, 261, 469, 409, 93$

As the above results show, using 16 intervals, the Simpson's rule gives the closest answer.

Three-point Gauss-Kronrod quadrature

$$x_1 = -\sqrt{\frac{3}{5}}, w_1 = \frac{5}{9}, x_2 = 0, w_2 = \frac{8}{9}, x_3 = \sqrt{\frac{3}{5}}, w_3 = \frac{5}{9}$$

$n = 1$
For [-1.5,1.5],

$$u = \frac{\beta - \alpha}{2} = \frac{1.5 - (-1.5)}{2} = \frac{3}{2} = 1.5 \text{ and } v = \frac{\alpha + \beta}{2} = \frac{-1.5 + 1.5}{2} = \frac{0}{2} = 0$$

$$\frac{3}{2} * \left(\frac{5}{9} f \left(-\frac{3}{2}\sqrt{\frac{3}{5}} \right) + \frac{8}{9} f \left(\frac{3}{2} * 0 \right) + \frac{5}{9} f \left(\frac{3}{2}\sqrt{\frac{3}{5}} \right) \right) = 2.162, 669, 113, 320, 397, 139, 19$$

$n = 2$
For [-1.5,0],

$$u = \frac{\beta - \alpha}{2} = \frac{0 - (-1.5)}{2} = \frac{1.5}{2} = 0.75 \text{ and } v = \frac{\alpha + \beta}{2} = \frac{-1.5 + 0}{2} = \frac{-1.5}{2} = -0.75$$

$$0.75 * \left(\frac{5}{9} f \left(-0.75\sqrt{\frac{3}{5}} - 0.75 \right) + \frac{8}{9} f (0.75 * 0 - 0.75) + \frac{5}{9} f \left(0.75\sqrt{\frac{3}{5}} - 0.75 \right) \right)$$

For [0,1.5],

$$u = \frac{\beta - \alpha}{2} = \frac{1.5 - 0}{2} = \frac{1.5}{2} = 0.75 \text{ and } v = \frac{\alpha + \beta}{2} = \frac{0 + 1.5}{2} = \frac{1.5}{2} = 0.75$$

$$0.75 * \left(\frac{5}{9} f \left(-0.75\sqrt{\frac{3}{5}} + 0.75 \right) + \frac{8}{9} f (0.75 * 0 + 0.75) + \frac{5}{9} f \left(0.75\sqrt{\frac{3}{5}} + 0.75 \right) \right)$$

$$\Rightarrow \text{Area} = 1.995, 002, 164, 031, 452, 427, 06$$

With only two intervals, the Gauss-Kronrod quadrature found the answer to almost the same accuracy as that of the Simpson's rule with sixteen intervals.