International Baccalaureate

Extended essay cover

Candidates must complete this page and then give this cover and their final version of the extended essay to their supervisor.

Candidate session number	[redacted]		
Candidate name	[redacted]		
School number	[redacted]		
School name	[redacted]		
Examination session (May or November)	MAY	Year	2012

Diploma Programme subject in which this extended essay is registered: PHYSICS

(For an extended essay in the area of languages, state the language and whether it is group 1 or group 2.)

Title of the extended essay: INVESTIGATING A WATER ROCKET

Candidate's declaration

This declaration must be signed by the candidate; otherwise a grade may not be issued.

The extended essay I am submitting is my own work (apart from guidance allowed by the International Baccalaureate).

I have acknowledged each use of the words, graphics or ideas of another person, whether written, oral or visual.

I am aware that the word limit for all extended essays is 4000 words and that examiners are not required to read beyond this limit.

This is the final version of my extended essay.

Candidate's signature: [redacted] Date: 13.01.2012

Supervisor's report and declaration

The supervisor must complete this report, sign the declaration and then give the final version of the extended essay, with this cover attached, to the Diploma Programme coordinator.

Name of supervisor (CAPITAL letters) ______

Please comment, as appropriate, on the candidate's performance, the context in which the candidate undertook the research for the extended essay, any difficulties encountered and how these were overcome (see page 13 of the extended essay guide). The concluding interview (viva voce) may provide useful information. These comments can help the examiner award a level for criterion K (holistic judgment). Do not comment on any adverse personal circumstances that may have affected the candidate. If the amount of time spent with the candidate was zero, you must explain this, in particular how it was then possible to authenticate the essay as the candidate's own work. You may attach an additional sheet if there is insufficient space here.

The topic was determined by the candidate's fascination by rockets. His extensive background study on rockets in general had been an ongoing engagement, the essay was just one further impetus. The water rocket provided an experimentally approachable form of rocket propulsion that lent itself to effective treatment of a research question.
In the preparatory reading phase the candidate had to face internet based sources being diverse in quality. He needed to reject some of them as scientifically unreliable, but "serious" sources provided conflicting information, too. In our discussions, the candidate demonstrated a sound critical approach to such sources.
The topic required the synthesis of mechanics and thermal physics and areas beyond the DP syllabus (quantitative treatment of an adiabatic change, equation of continuity, Bernoulli's law). In all this theoretical background, as well as in homogeneous and inhomogeneous differential equations, he was tutored by friends outside the school.
As reflected by my interviews with him, he successfully developed an understanding in these areas.
The candidate's writing is somewhat hard to follow (same is the case with his tests and lab reports): He cannot be convinced that he should not talk about something until he defined/stated what it was. It needed considerable effort on my part to question him about all the content of the essay, and gain evidence that he thoroughly understands everything. He does. (The final version of the essay became a lot more readable than the first draft was.)

This declaration must be signed by the supervisor; otherwise a grade may not be issued.

I have read the final version of the extended essay that will be submitted to the examiner.

To the best of my knowledge, the extended essay is the authentic work of the candidate.

I spent 4 hours with the candidate discussing the progress of the extended essay.

Supervisor's signature: ______ Date: 1 March 2012

Assessment form (for examiner use only)

Candidate session number	[redacted]

Achievement level

Criteria	Examiner 1	maximum	Examiner 2	maximum	Examiner 3
A research question clear + focused	2	2	2	2	
B introduction Theory in wrong place	1	2	2	2	
C investigation	4	4	4	4	
D knowledge and understanding	4	4	4	4	
E reasoned argument	4	4	4	4	
F analysis and evaluation — all so good	3	4	4	4	
G use of subject language	4	4	4	4	
H conclusion unresolved issues?	1	2	1	2	
I formal presentation	4	4	4	4	
J abstract	2	2	2	2	
K holistic judgment one of the very best I have seen.	4	4	4	4	
Total out of 36	33		35		

ne of examiner 1: [redacted] Examiner number: [redacted]
PITAL letters)

ne of examiner 2: [redacted] Examiner number:
PITAL letters)

ne of examiner 3: ______ Examiner number: ______
PITAL letters)

IB Cardiff use only: B: ✓

IB Cardiff use only: A: 094329 **Date:** 18/5

EXTENDED ESSAY

is one of the best I have ever seen.

INVESTIGATING A WATER ROCKET

An ideal subject for a Physics EE & a well chosen & well focused RQ. The practical investigation is outstandingly well done, with some very clever techniques employed! Figure 9 is the best I have ever seen in an EE. The theory development is ambitious & not without mistakes – but these do not spoil the overall excellence of this piece. A little disappointing is the constant referral to "$\frac{1}{3}$rd or about 40 or 50%". The agreement between theory and experiment does seem a little too good to be true. Uncertainties in the final volume ratio are underestimated. Overall a very impressive piece of work. Nearly all of my initial concerns were addressed by the author. Clearly a capable, creative & dedicated student. Very well done!

Name:

Session number:

School:

Subject: Physics

Supervisor's name:

Word counts: 3990

Abstract

A self-made water rocket is an inverted plastic bottle which is filled with a fixed amount of pure water. Air is pumped inside it until the pressure of the trapped air is large enough to be able to push the plug out.

The motion of the water rocket was investigated experimentally, with the use of a high-speed camera and a video analysis software. The purpose of the investigation was to determine the optimal amount of water (fuel) needed for the rocket to fly as high as possible, and to provide the explanation of the results based on the principles of physics. RQ

The motion of the water rocket can be divided into three stages: the rocket propelled by the thrust of the ejected water, the further rise of the empty rocket to the highest point of the trajectory and the fall of the rocket. A mathematical model was set up for each of the first two stages, and the values of rising times, velocities and heights calculated from the model were compared to the experimental results. S

According to several articles and water rocket simulators the ideal amount of water is about one third or between 40% and 50% of the total capacity of the body of the inverted plastic bottle. It means that the water rocket will fly the highest if one third or 40% to 50% of the bottle is occupied by water. The statements were neither justified by the model nor the experiment. However, the model and the experimental results fit well, which means the model is reliable, therefore the statements are only acceptable as a rough approximation. C

Good start

Word count: 265

Contents

I. INTRODUCTION

I have always been interested in space shuttles so I did some investigation to reveal the physics behind the process of using rockets to launch the space shuttle. Rockets apply Newton's second and third law by creating thrust and do not require external forces but fuel inside them to be launched. A water rocket applies the same physical laws and it is easier to launch than a space shuttle, moreover, a water rocket is very much similar to a NASA rocket [1]. A water rocket is an inverted plastic bottle which is filled with a fixed amount of water. Air is pumped inside it until the pressure of the trapped air is large enough to push the plug out. The thrust of the water squirting out propels the rocket (Figure 2). However, no matter how much fuel is filled into the water rocket, it cannot be launched high enough to set it in orbit. how low!

Research Question:

How does the maximum height that the water rocket can reach change with the variation of the initially filled amount of water? is water the fuel? ? what about the compressed air?

RQ clear + focused

It seems reasonable that there is an optimal amount, since in the case of not filling the bottle enough, the rocket will lack fuel. In the other case when the bottle is overfilled, the rocket will be too heavy and not uses up all of its fuel. According to several articles and water rocket simulators the ideal amount of water is about one third [2, 3, 4] or 40% to 50% [5, 6] of the total capacity of the body of the rocket. It means that the water rocket will fly the highest if one third or 40% to 50% of the bottle is occupied by water. good

??? $\frac{1}{3} = 33\%$

The goal of my investigation was to find out which hypothesis is proved or disproved by experiment, and to justify the results by the principles of physics.

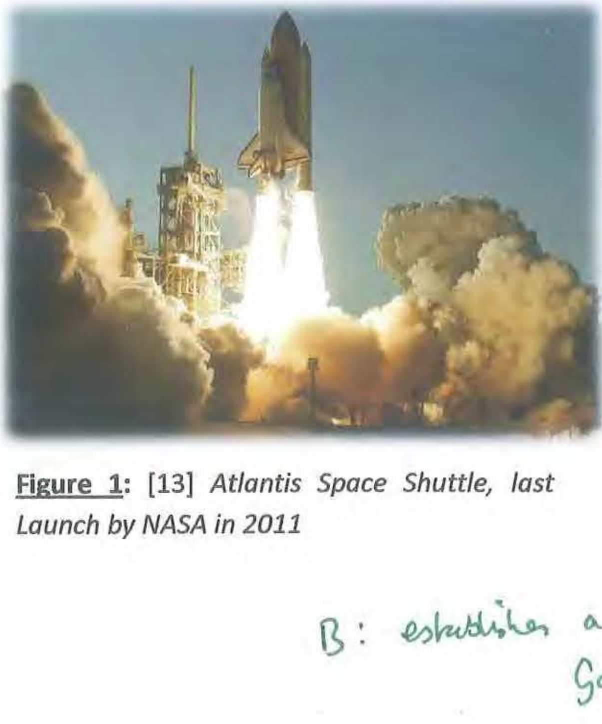


Figure 1: [13] *Atlantis Space Shuttle, last Launch by NASA in 2011*

Figure 2: *The investigated water rocket, first launch by me in 2011*

poor quality image

B: establishes a disagreement amongst research. Good enough reason to study it!

a) Constructing a water rocket

In the experiment one of the simplest versions of a water rocket was made. This particular type of water rocket is a simple PET plastic bottle that has two heads. Therefore, there needs to be two PET bottles. One bottle's top is going to be the nose cone; which is attached to the other bottle's bottom (Figure 3). This enhances the stability of the rocket and reduces air drag while it flies.

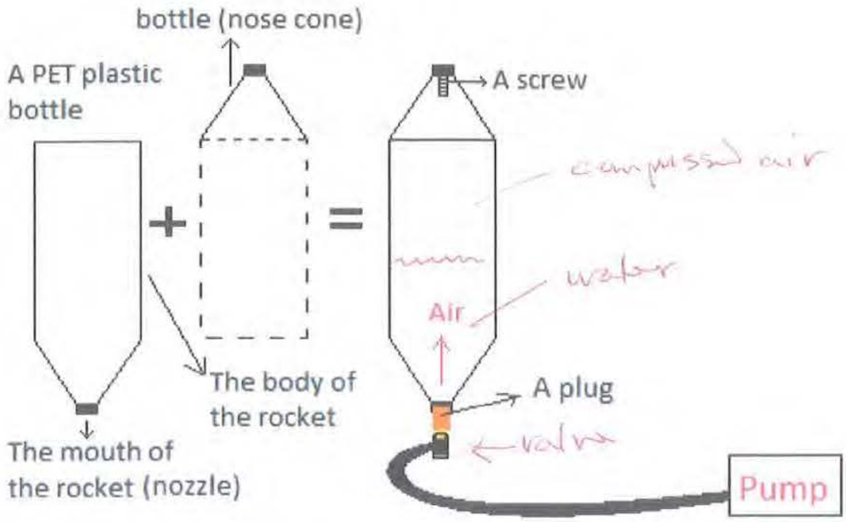


Figure 3: *The construction of the water rocket*

In the experiment, the body of the rocket was an old 2-litre coke bottle. Nowadays, to protect the environment only thin walled PET bottles are produced. To make the rocket's flight more stable the harder or older type of bottles should be used.[1] During the experiment, it was experienced that the flight of the rocket was not really straight so a little mass (a screw) was placed in the nose cap in order to straighten the flight. Just like the darts that have a little mass on the nose to make their flight more stable. Moreover, if the mass of the rocket is to be varied then nuts can be put on the screw to make it heavier.

To operate the rocket, at first, a fixed amount of water has to be filled in and then air is continuously pumped in it until no more can be added. This means that the internal pressure from the bottle is great enough to push out a plug and so, the water. Finally, the rocket rises due to the ejection of water and the trapped air.

plug friction very difficult to keep constant between trials

However, the plug is not a regular cork since not only water but also air has to be added into the bottle. This is why the cork is drilled longitudinally and centrally to be able to insert a

[1] From the experiment, it could be seen that the flight of a thin walled water rocket was very much unstable (not straight) than the older type. The older type mostly landed within a circle with a radius of three meter centered at the launch pad.

bicycle tube valve. The cork was too long for the valve so it had to be shortened with a knife. This method allows adding air into the bottle without the water flowing out.

Figure 4: *The stopper used in the mouth of the rocket in the experiment*

Finally, to pump air, the valve has to be connected to an air supply. Once the air is pumped in, due to the expansion of the compressed air the cork is pushed out which will result in the ^ of water acceleration of the rocket upwards. The easiest and cheapest way to pump in air is to use bicycle pump or car tire foot pump. It is also important that the pump has to have a pressure meter for it should be known with what initial pressure it is launched. In this experiment, a bicycle pump was used which was equipped with a pressure gauge with the smallest scale unit of one fifth of a bar. This is why the measurement of pressure was the least precise relative to other quantities measured in the experiment. See p12 for theory (wrong place for it)

b) Launching the water rocket

The launcher has two important parts. The wooden base supports all the other components and provides stability. The launch tube: helps the rocket fly vertically, especially at the beginning of the launch. The launch tube was cut from a PVC tube, fixed to the base with wires (Figure 5).

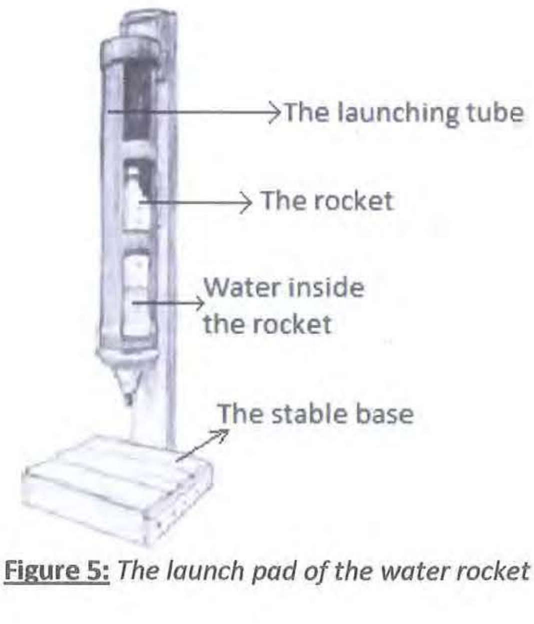


Figure 5: *The launch pad of the water rocket*

II. THE EXPERIMENT

a) The technical data of the water rocket

For measuring the diameter of the body of the rocket and the diameter of the nozzle a caliper was used. To measure the volume of the body of the rocket, the empty bottle was filled with water and then that amount of water was poured into a measuring cylinder which had centiliter divisions. The mass of the empty rocket was determined by putting the rocket on a kitchen scale on which the smallest unit was one gram.

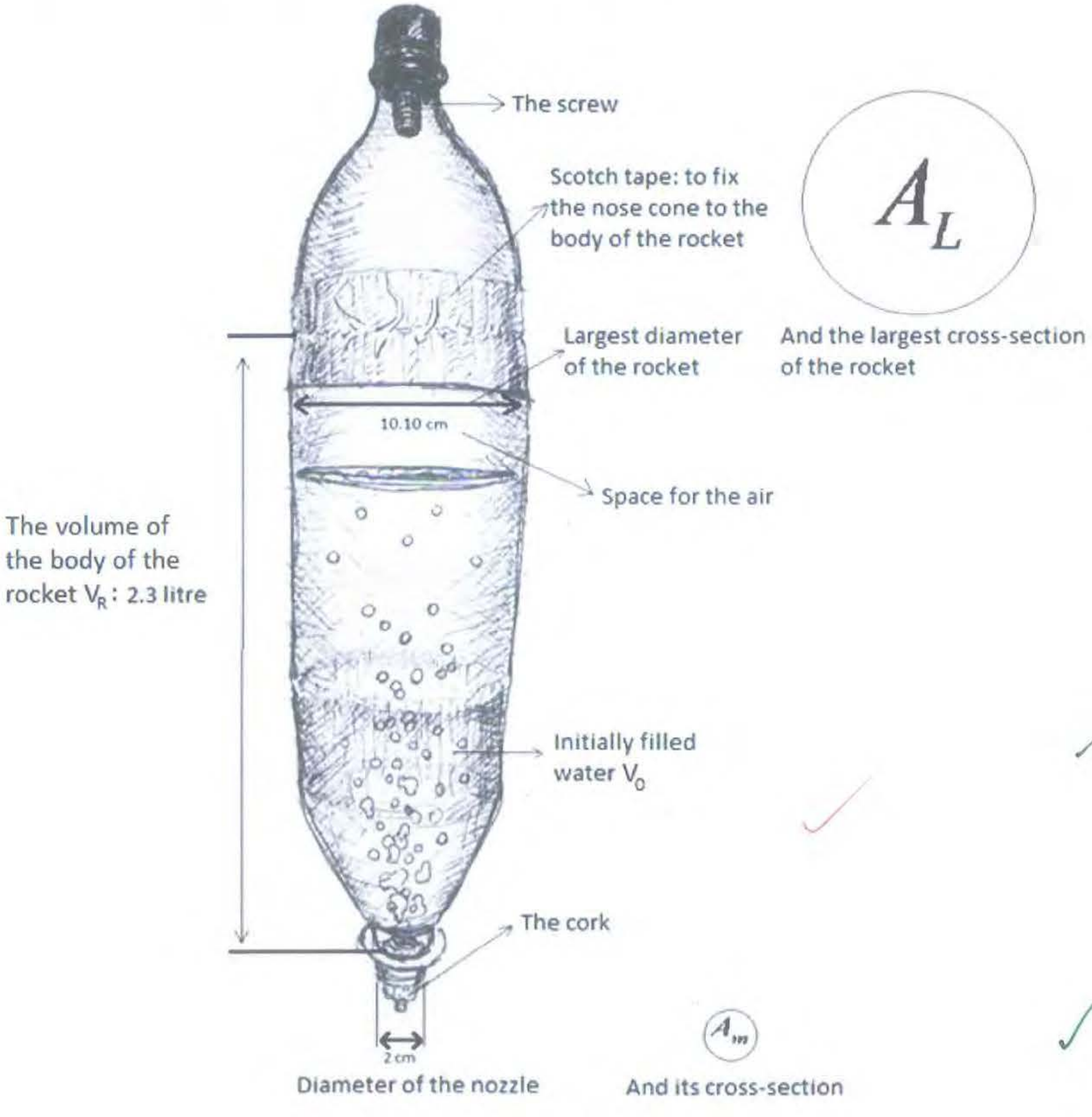


Figure 6: *The initial parameters of the water rocket*

- Largest diameter and the cross-sectional area of the rocket:

$$d_L = (10.10 \pm 0.03)\ \text{cm and}\ A_L = (80.1 \pm 0.5)\,\text{cm}^2$$

- Diameter and the cross-section of the nozzle:

$$d_m = (20.0 \pm 0.2)\ \text{mm and}\ A_m = (3.14 \pm 0.06)\,\text{cm}^2$$

- Volume of the body of the rocket:

$$V_R = (2300 \pm 5)\,\text{ml}$$

- Mass of the empty rocket :

$$m_R = (225 \pm 1)\,\text{g}$$

Note: The errors of the measured values are the half of the smallest scale unit but if the measurement is dubious then larger error should be used.

In the case of the error of the cross-sectional areas the error percentages are added. For instance, $A_m = \left(\frac{d_m}{2}\right)^2 \times \pi$ will have an error of 2% because d_m has 1% error and its square will have an error of $2 \times 1\% = 2\%$.

b) What was measured?

Eleven different amounts of water were used. To match the hypothesis about $\frac{1}{3}$ or 40% to 50% volume, the volume of the rocket was divided into twelfths and the multiples of it were measured with the same volume meter as above, with centiliter divisions. The fractions of the body of the rocket filled with water are shown:

$\frac{0.0}{12}$	$\frac{2.0}{12}$	$\frac{2.5}{12}$	$\frac{3.0}{12}$	$\frac{3.5}{12}$	$\frac{4.0}{12}$	$\frac{4.5}{12}$	$\frac{5.0}{12}$	$\frac{5.5}{12}$	$\frac{6.0}{12}$	$\frac{8.0}{12}$
0	0.17	0.21	0.25	0.29	0.33	0.38	0.42	0.46	0.50	0.67

Table 1: *The ratios of the water and the body of the rocket that were examined*

The movement of the water rocket can be divided into three stages. The first stage is when the rocket ejects water and accelerates to its maximum velocity. This stage ends when all the water has left the bottle, then, the second stage begins; in which the rocket still goes up without ejecting any water and then it stops due to air drag and gravitational force. The last phase is when the rocket falls down to the ground. The investigation only deals with the first two stages.

i. The application of the camera

The most convenient way to analyze the motion of the water rocket is to use a high-speed camera, because the rocket ejects all of its water in tenth of seconds. Mostly, ordinary camcorders record 30 frames per second (fps), while a high-speed camera can record much more frames. In the experiment, a Casio Zr100 type of camera was used which was able to record in 1000fps, 480fps, and 240fps. With the help of this camera, the displacement of the rocket with respect to time was determined. Although the more frames it uses, the easier it can be analyzed, however, using slow motion recording mode will decrease the quality of the video. Therefore, it was suitable to use the 240fps mode. Unfortunately, only the first stage of the rocket's movement was recorded because at larger heights the error of perspective is increasing.

It should also be noted that the camera should be used on a stand; otherwise the camera will be moving and it will affect the measurements in the video analyzer.

Figures *7* and *8* show the time of ejection of the rocket; it can be seen that the water runs out at *0.125* s and after that only the water spray leaves the bottle.

X

clearly not 1.875 s gap!!

Figure 7: *A water rocket, when it has ejected all of its water*

Figure 8: *A water rocket, just after it ejected all of its water*

Good

Figure *9* shows another launch's every fifth frame with the very top of the rocket marked with a red circle.

error

great.

Figure 9: *Shows every fifth frame of a launch of a water rocket with 767ml of water in it*

absolutely excellent!

ii. The analysis of the recorded video

The recorded video was analyzed by a video analyzer program, Logger Pro 3.8.4. With the help of this program, the maximum velocity of the rocket as well as the time taken to eject all of its water was determined. During this experiment every second or fifth frame of the video was examined (Figure 10).

Figure 10: *The method of examining every second frame in Logger Pro 3.8.4. The red line is a reference distance which is 0.58 m.*

Note: At larger initial water amount, collecting data points from every fifth frame allowed a nicer graph.

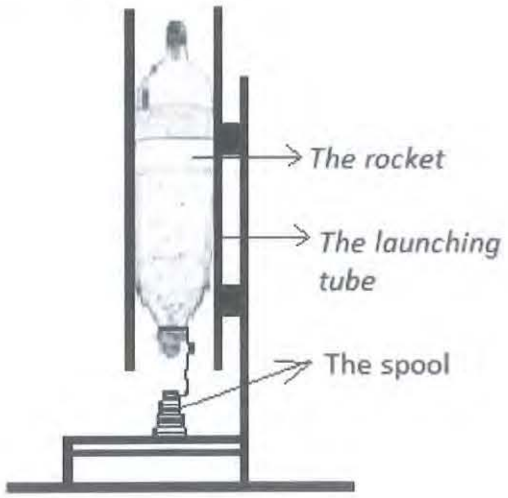
iii. The initial pressure

The pressure of air pumped in the bottle was read from the pressure meter while continuously adding air into the rocket. The maximum initial pressure is shown on the pressure meter when the plug in the mouth of the rocket has just been pushed out. This variable had to be controlled by inserting the cork into the mouth of the bottle with the same pushing force and same circumstances. For instance, if the cork was initially inserted when it was dry then before all the launches the cork as well as the inner mouth of the bottle should be dried. Unfortunately, the initial pressure could not be controlled very well, the initial pressures had large deviations.

iv. The maximum height

The maximum height reached by the rocket was measured by using a thread. The spool was placed right under the rocket and the end of the thread was fastened to the mouth of the rocket. Therefore, when the rocket rises upwards it pulls the thread after itself until it reaches the highest point of its trajectory then it falls down. Afterwards, the pulled out thread was measured by measuring tape. It would be more comfortable to determine the height from the video but as mentioned before, only the first stage could be recorded since the rocket still goes upwards after it has ejected all the water and it gets out of view of the camera.

Figure 11: *The set up of the height measuring equipment*



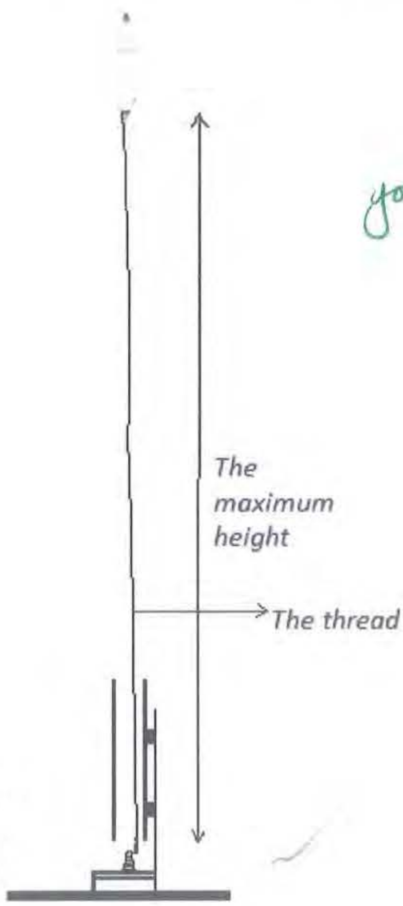


Figure 12: *The determination of the maximum height reached by the rocket*

good.

c) The results of the measurements

Figures *13* and *14* produced by Logger Pro, are examples to illustrate the result of collecting data points from every fifth and second frame from a recorded video.

a)

b)

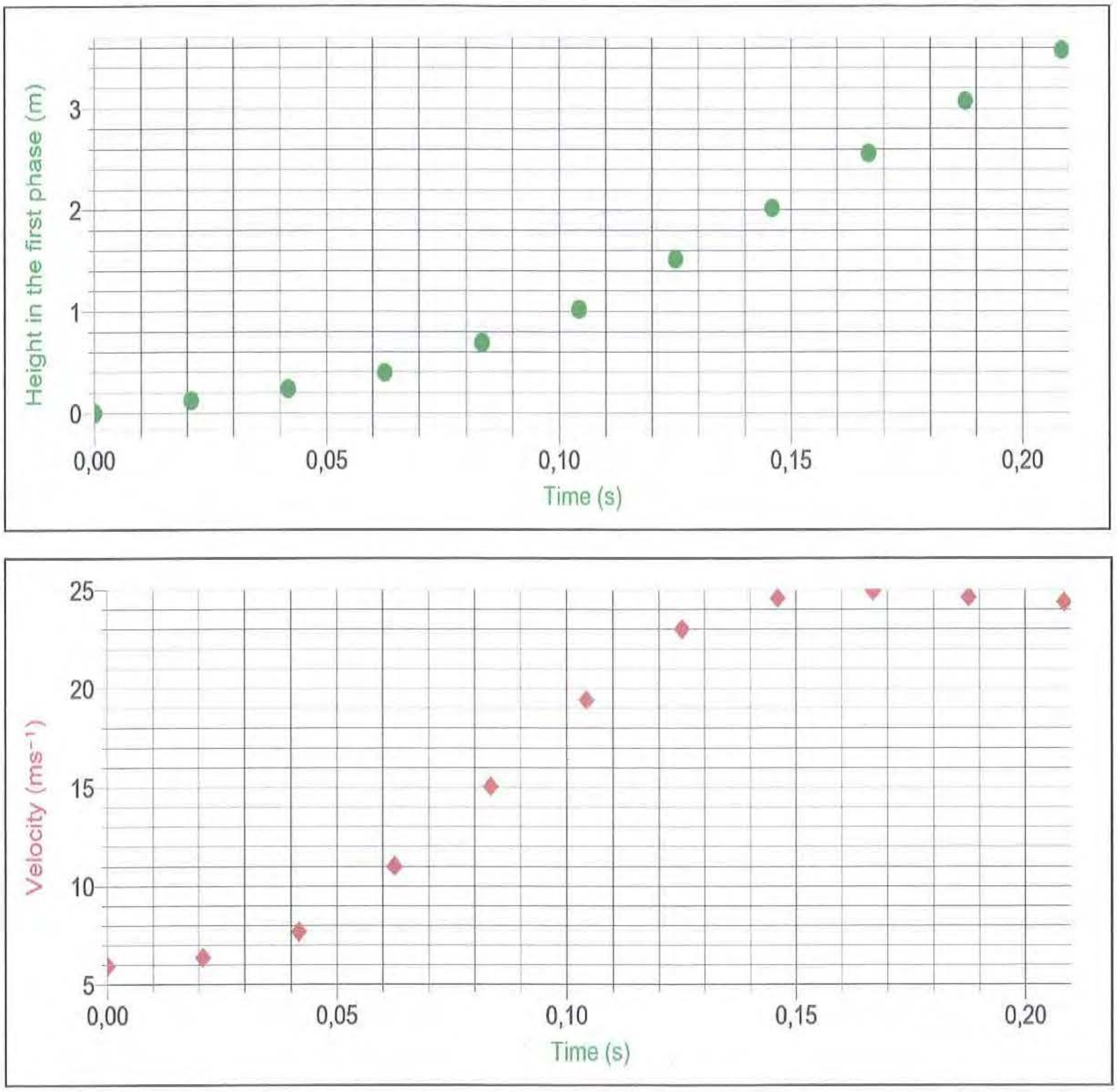


Figure 13: *Shows (a) the displacement and (b) the velocity of a 575ml rocket with respect to time when collecting data from every fifth frame*

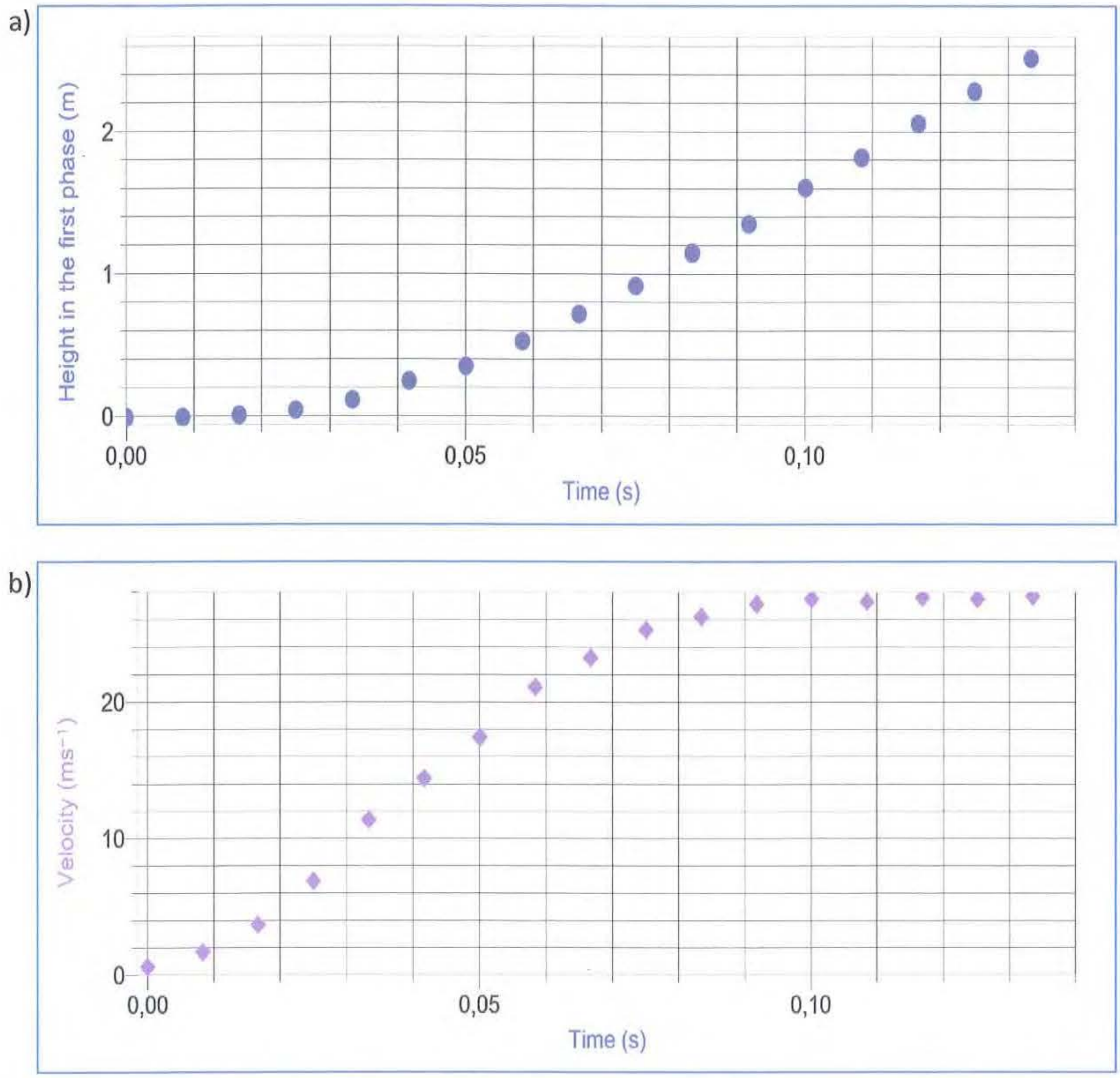


Figure 14: *Shows (a) the displacement and (b) the velocity of a 383ml rocket with respect to time when collecting data from every second frame*

Images show only fraction of the rocket – It would be a good idea to show one example of the complete flight of the rocket

The following table shows the average measured values of the experiment and the errors were the difference of the average and the measured value:

Understandably larger uncertainty of ~ 10%

The average initial pressure p_1: (2.9 ± 0.3) bar[2]				
The volume ratio of the water and the bottle	Initial water amount V_0 (ml)	Measured maximum velocity v_{max} (ms^{-1})	Measured time of ejection of water t_t (s)	Measured maximum height h_{max} (m)
	error: $\pm$5ml	error:$\pm 1.1\ ms^{-1}$	error: 0.003	error: $\pm$0.5m
$^{2.0}/_{12} = 0.17$	383	20.1	0,070	16.2
$^{2.5}/_{12} = 0.21$	479	22.5	0,088	18.6
$^{3.0}/_{12} = 0.25$	575	25.0	0,088	19.7
$^{3.5}/_{12} = 0.29$	671	24.4	0,142	21.9
$^{4.0}/_{12} = 0.33$	767	26.0	0,129	22.8
$^{4.5}/_{12} = 0.38$	863	25.5	0,167	23.1
$^{5.0}/_{12} = 0.42$	958	25.1	0,186	23.6
$^{5.5}/_{12} = 0.46$	1054	24.1	0,222	21.9
$^{6.0}/_{12} = 0.50$	1150	20.0	0,267	17.8
$^{8.0}/_{12} = 0.67$	1533	9.8 (not exact)	-	8.5

• MAX.

Table 2: *Shows the results of the measured variables in the experiment*

This should be graphed!!

The values of v_{max} and t_t at the $\frac{8.0}{12}$ water amount are not exact for the rocket contained too much water and the camera could not record the whole phase of ejection.

Looks like no repeats were done to reduce the uncertainty due to the release pressure. Pity

[2] Unfortunately, the initial pressure could not be controlled as a constant very well, so an average had to be taken.

III. ANALYZING AND MODELING THE PHASES OF THE ROCKET'S MOVEMENT

this [illegible] section should come earlier

a) The phase when the rocket accelerates

At first, to estimate the rocket's maximum height for a particular amount of water, the maximum velocity should be obtained. Theoretically, the acceleration of the rocket ends when the fuel runs out which means the rocket reaches its maximum speed at the end of the first stage.

i. Expressing the velocity of the water and the time of ejection of the water

While the air is continuously pumped into the closed bottle, the pressure of the trapped air is increasing. Thus, the cork will be pushed out when the sum of the forces of the pressure of the compressed air and the hydrostatic pressure of the water acting on the inner surface of the cork is greater than that of the atmospheric pressure acting on the external end of the cork and the maximum of the static friction between the side of the cork and the wall of the mouth of the bottle together (Figure 15).

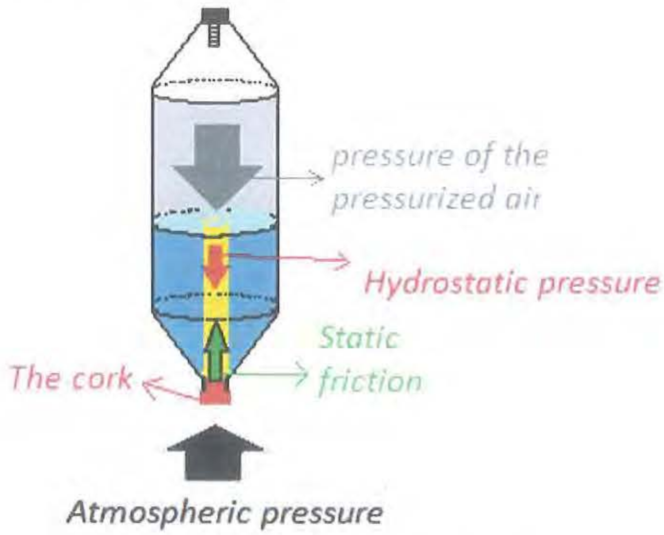


Figure 15: *The water rocket before it is launched*

After the cork is pushed out, the compressed air expands really fast and pushes out the total or only certain amount of water from the bottle. The process takes place really fast (e.g.:*0.125* s) which means it is a good approximation to assume that the gas and its surroundings do not exchange heat. This process is called adiabatic expansion, and the Poisson equation relates the different states of the gas [7, 8, 10]:

$$p_1 V_1^{\gamma} = p(t)[V(t)]^{\gamma} \tag{1}$$

where p_1, V_1 and $p(t)$, $V(t)$ are the pressure and the volume initially, and at a later time *t*. Now $\gamma = 1.4$ because air mostly consists of diatomic gas [7, 10].

but what is to be done with this eq?

Truly, the water in the bottle does not necessarily leave completely. This is why there are two cases. Provided that the initial water level is not high and enough air is added into the bottle, the compressed air will expand to greater volume than the volume of the rocket. Only this way can the compressed air decrease its pressure to the atmospheric pressure. In this case, all the amount of water will leave the bottle by the end of the ejection phase **(case 1)** [8].

In the event of loading too much water or if the compressed air's pressure is low enough, the air will expand to lower volume than the volume of the rocket. In this way, the air will not push all the water out and the rocket will fly with the remaining amount of water **(case 2)**. In the experiment, at the largest amount of water (*1.533* litre), the rocket still had water inside it after it landed. In order to launch the rocket as high as possible, the appropriate amount of water should be filled in where the rocket just uses up all the fuel. In this way, **case 1** should be examined.

Good observations

THE FIRST CASE

The water is continuously leaving the bottle, thus, applying the law of continuity for incompressible liquids [8, 10]: the same volume of water is pushed downwards by the air, as the volume coming out.

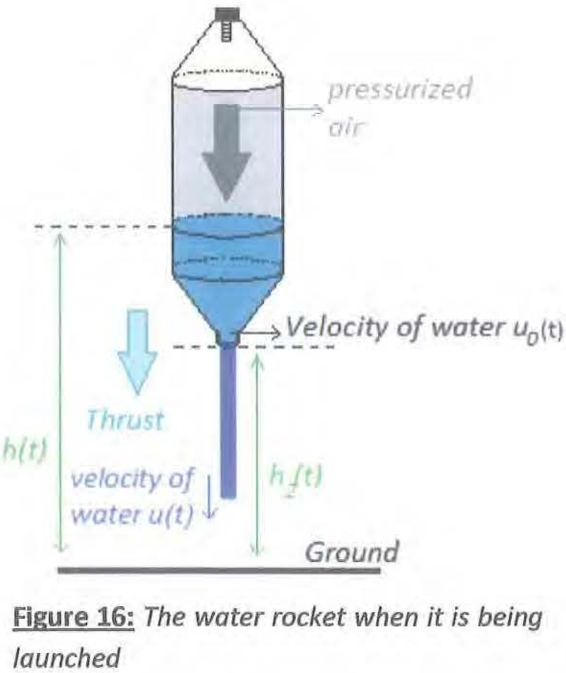


Figure 16: *The water rocket when it is being launched*

Owing to the laws of continuity and energy conservation, the acting pressures and speeds at the two ends of the water column at a time t are connected by Bernoulli's principle [7, 8, 10]:

$$p(t)+\rho_w g h(t)+\frac{1}{2}\rho u_0(t)^2 = p_a+\rho_w g h_2(t)+\frac{1}{2}\rho_w u(t)^2$$

reference?

where $p(t)$ and $p_a = 10^5\,Pa$ are the pressures of the compressed air at time t and the pressure of the atmosphere, $\rho_w = 1000\frac{kg}{m^3}$ is the density of the water, $h(t)$, $h_2(t)$ are the heights of the water surfaces, and $u_0(t), u(t)$ are the velocity of the water relative to the bottle at the water surfaces inside and outside of the bottle at time t. OK

Since the inner cross-sectional area is much greater than that of the mouth, it can be assumed that the velocity of the water $u_0(t)$ is $0\,\mathrm{ms}^{-1}$. Actually, the inner cross sectional area is 80.1 ± 0.5, and the mouth is 3.1 which is not really negligible, but considering the inaccuracy of the pressure reading this step is justified.

Since the hydrostatic pressures $\rho_w g h_1(t)$ and $\rho_w g h_2(t)$ are really small[3] compared to the pressures $p(t)$ and p_a, they can also be neglected. So the simplified equation is:

$$p(t) = p_a + \frac{1}{2}\rho_w u(t)^2$$

Hence, with the right arrangements and the omission of the mentioned quantities, the velocity of the water $u(t)$ relative to the rocket at a given time t is [8]:

$$u(t) = \sqrt{\frac{2(p(t)-p_a)}{\rho_w}} \tag{2}$$

This is a simpler result than that in [7].
The pressure at time t can be expressed by applying equation (1). So, the initial velocity of the water u_1 can be obtained by substituting the measured initial pressure

$$u_1 = \sqrt{\frac{2(p_1 - p_a)}{\rho_w}}$$

[3] If the water column above the inner surface of the cork is assumed to be 1 litre (which is a little bit of an exaggeration) then the hydrostatic pressure is: 0.00314 bar, which is two orders of magnitude smaller than the error of the pressure readings.

Consider time t, when all the water went out. Therefore, using equation (1) $V(t)$ equals the volume of the rocket V_R, and the pressure at the end of the first stage p_2 is:

$$p_2 = \left(\frac{V_1}{V_R}\right)^{\gamma} p_1 \qquad (3)$$

<u>Note</u>: $V_1 = V_R - V_0$

Hence, the final velocity u_2 of the water coming out can be calculated by substituting this value of the final pressure p_2 in (2). For instance, for $V_0 = 0.383$ litre,

$$p_2 = \left(\frac{1.917}{2.300}\right)^{1.4} \times 2.9 = 2.3\,\text{bar} \qquad u_2 = \sqrt{\frac{2(2.3-1)\times 10^5}{1000}} = 16.1\,\text{ms}^{-1}$$

After obtaining the initial and the final velocity of the water in the ejection phase, an estimation of average velocity can be used by taking the average of the initial and final velocity of the water, which means the estimation assumes a uniform acceleration. – this is an approximation to avoid integral calculus ✓ fine!

Another assumption should be considered: the water column that comes out, which is exactly the volume of the initially used water V_0, can be regarded as a cylinder which base is the cross-section of the mouth of the bottle, A_m.

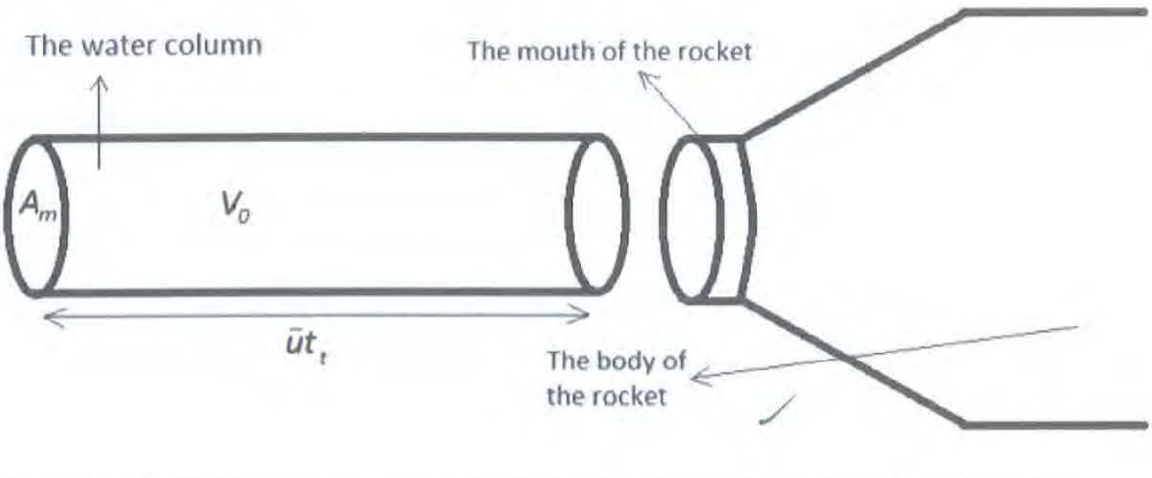


Figure 17: *The equation of continuity: the water leaving the rocket has a form of a cylinder*

Where $V_0 = A_m \bar{u} t_t$, $\bar{u} = \frac{u_1 \pm u_2}{2}$ and t_t is the time taken to eject all the water. In this way, the time taken for the rocket to push all the water out can be expressed as:

water volume

$$t_t = \frac{V_0}{\bar{u}A_m}$$

area of bottle x-section

mean velocity

➢ ***The relative velocity and the time of ejection of the water***

The following table compares the calculated values based on the above formulae to the measured values taken from table 2:

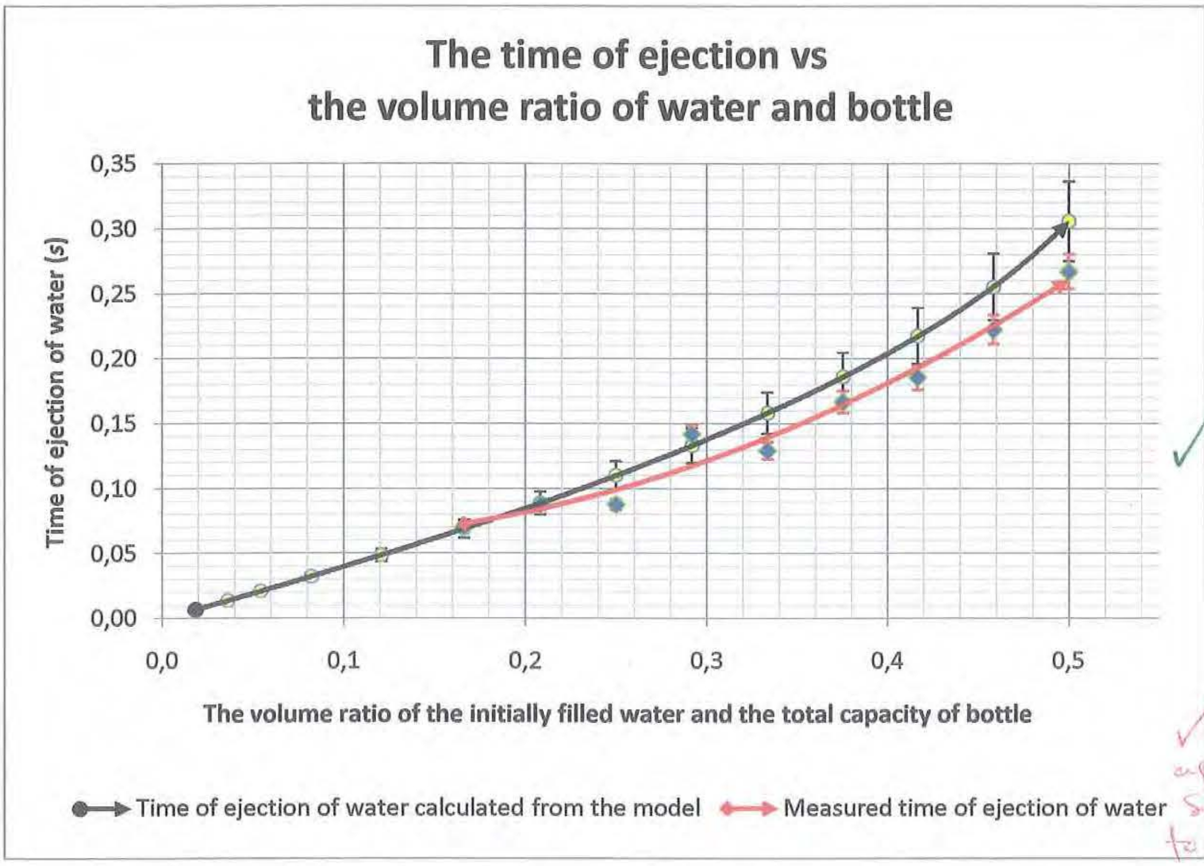
Table 3: *Shows the variables needed to determine the calculated value of the time of ejection of water*

Initial parameters						
Average initial pressure p_1: (2.9 ±0.3) bar			Initial relative water velocity $u_1 = (19.5 \pm 2.0)\ \mathrm{ms^{-1}}$			
Cross-sectional area of nozzle $A_m = (3.14 \pm 0.06)\ \mathrm{cm^2}$						
Volume ratio of water and the bottle	Amount of water V_0 (*l*)	Calculated values from model (end of stage 2)				Measured time t_t (s)
		Final pressure p_2 (bar)	Final water velocity u_2 ($\mathrm{ms^{-1}}$)	Average of u_1 and u_2, $\bar{u}$($\mathrm{ms^{-1}}$)	Ejection time t_t (s)	
-	error: ±0.005	error: $\pm\approx 10\%$	error: $\pm\approx 10\%$	error: $\pm\approx 10\%$	error: $\pm\approx 10\%$	error: ±0.003
$^{2.0}/_{12} = 0.17$	0.383	2.2	15.8	17.6	0.07	0.070
$^{2.5}/_{12} = 0.21$	0.479	2.1	14.8	17.1	0.09	0.088
$^{3.0}/_{12} = 0.25$	0.575	1.9	13.7	16.6	0.11	0.088
$^{3.5}/_{12} = 0.29$	0.671	1.8	12.6	16.0	0.13	0.142
$^{4.0}/_{12} = 0.33$	0.767	1.6	11.3	15.4	0.16	0.129
$^{4.5}/_{12} = 0.38$	0.863	1.5	10.0	14.8	0.19	0.167
$^{5.0}/_{12} = 0.42$	0.958	1.4	8.5	14.0	0.22	0.186
$^{5.5}/_{12} = 0.46$	1.054	1.2	6.8	13.1	0.26	0.222
$^{6.0}/_{12} = 0.50$	1.150	1.1	4.4	12.0	0.31	0.267
$^{8.0}/_{12} = 0.67$	1.533	0.6	-	-	-	-

boundary between case1 / case2

Note: In the case of p_2, (3) is used to calculate the error. The error percentage of the values of the fraction and p_1 has to be added because the equation contains division and multiplication. Since the error of V_1 and V_R are small compared to the error of p_1, it can be neglected so the error of p_2 is that of p_1, which is 10%.

It can be seen that the model of the first case works up to *1.533 litre,* where the final pressure becomes less than 1 bar and using equation (2) would make no sense[4]. From that V_0 onwards, **case 2** prevails.



agreement shows to seem to be true.

Figure 18: *Shows the calculated and the measured value of the time of ejection of water at the initial pressure of 2.9 bar*

Case 1 (to find v)

ii. The motion of the rocket

To describe the motion of the rocket in this phase, Newton's second law is applied [7, 12]:

$$F = \frac{dP}{dt}$$

[4] If the final pressure is smaller than 1 bar, then there would be a negative number under the square root.

At a time t after the launch the mass of the rocket is $m(t)$ and its velocity relative to the ground is $v(t)$. The total momentum of the system (bottle plus water inside) at time t is:

$$P(t) = v(t)m(t) \tag{4}$$

As water is continuously ejected from the bottle the mass of the rocket is continuously decreasing with respect to time, thus, the lost mass Δm is a negative quantity. During a small time period, Δt elapsed, the lost mass $|\Delta m|$ of water leaves the bottle with the velocity of $\bar{u}$ relative to the bottle, which then has a velocity of $-[\bar{u} - v(t)]$ relative to the ground. If the velocity of the rocket is $v(t)+\Delta v$ at time instant $t+\Delta t$. The total momentum of the rocket is:

$$p(t+\Delta t) = (v(t)+\Delta v)(m(t)+\Delta m), \tag{5}$$

and the momentum of the ejected water is:

$$p_w(t+\Delta t) = -[\bar{u} - v(t)] \times |\Delta m| = \Delta m[\bar{u} - v(t)] \tag{6}$$

Since, the momentum of the system after time $t+\Delta t$ is equal to the momentum of the rocket and the ejected water at that time, $\Delta P = p(t+\Delta t) + p_w(t+\Delta t) - P(t)$

If equations (4)(5)(6) are substituted into Newton's second law, after the simplifications the equation[5]

$$F = \lim_{\Delta t \to 0}\left(\frac{m(t)\Delta v + \Delta m \Delta v + \Delta m\bar{u}}{\Delta t}\right) \tag{7}$$

is obtained. At small Δt, the term $\Delta m \Delta v$ can be neglected. Therefore equation (7) can be simplified as:

$$F = \lim_{\Delta t \to 0}\left(\frac{\Delta v}{\Delta t}m(t) + \frac{\Delta m}{\Delta t}\bar{u}\right)$$

In this case, the external forces acting on the rocket are the gravitational force $-m(t)g$ and the air drag force $-Cv(t)$, where C is a constant. To express the velocity, only the gravitational force needs to be taken in account, because the air drag is small and it would not vary much[6] the maximum velocity:

[5] The deduction of the equation (7) is shown in the appendix.

[6] The air drag in this phase would slow the rocket by $0.048\ \mathrm{ms}^{-1}$ (calculated in the approximation of C section in appendix.

$$-m(t)g = \frac{dv}{dt}m(t) + \frac{dm}{dt}\bar{u} \tag{8}$$

again, approximate

Hence, after solving the equation (8) the velocity at any time t can be described by the Tsiolkovsky theorem [9]. However, a more complicated form (11) is used here than in [9], since this rocket had an initial velocity, for the video could not be examined exactly at the moment of the launch (because it could not be seen in the video), and the gravitational force was taken into account as well.[7]

$$v(t) = v_0 + \bar{u}\, ln\left[\frac{V_0\rho_W + m_R}{(V_0 - \bar{u}A_m t)\rho_W + m_R}\right] - g(t) \tag{11}$$

➤ *Calculating the maximum velocity of the rocket*

To determine the initial velocity v_0 of the rocket (11) should be fitted into the graph, made with Logger Pro (it shows data point from every fifth frame of a rocket with *767*ml water). The model only works until the end of the water ejection phase, which in this case is about $t_t = 0.150\,s$.

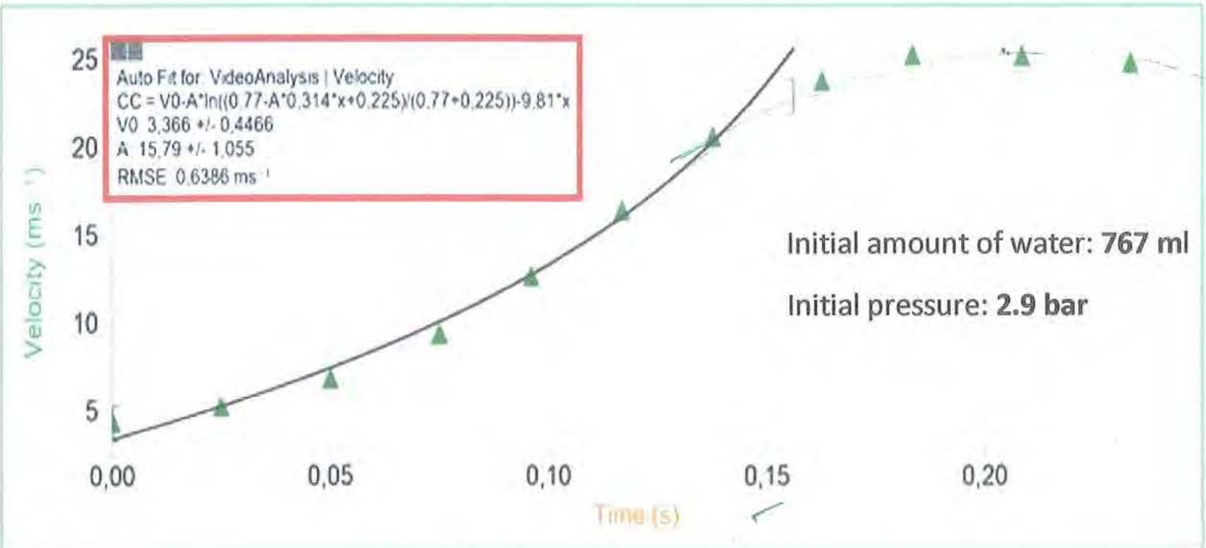


Figure 19: *Shows the application of Tsiolkovsky theorem (11) on the measured values of the maximum velocity in Logger Pro 3.8.4*

Given the constants of ρ_W, m_R, A_m, the program gives the values of the parameters $v_0 = (3.366 \pm 0.4466)\text{ms}^{-1}$ and $\bar{u} = (15.79 \pm 1.055)\text{ms}^{-1}$ (denoted by *A* in the red box), that is $v_0 = (3.4 \pm 0.5)\text{ms}^{-1}$, $\bar{u} = (16 \pm 1)\text{ms}^{-1}$. The values of $t_t = 0.150\text{s}$ and $\bar{u} = (16 \pm 1)\text{ms}^{-1}$ accord well with the calculated values of table 3 (shaded).

[7] The deduction of equation (11) is shown in the appendix.

If more data points are taken (e.g.: every second frame is examined), which means a more detailed measurement is done then the initial velocity v_0 is much smaller (close to 0), because originally the rocket was launched at rest. Thus, in the model of the maximum velocity v_0 is neglected.

The values of table 4 below are obtained by substituting the values of t_t (from table 3) into (11), using $v_0 = 0\ \mathrm{ms}^{-1}$.

Table 4: *Shows the maximum velocity and the final pressure after it has ejected all of its water*

Initial parameters					
Mass of empty rocket $m_R = (0.225 \pm 1)\mathrm{g}$			Density of water: 1000 $\frac{\mathrm{kg}}{\mathrm{m}^3}$		
Average initial pressure p_1: (2.9 ±0.3) bar			Area of nozzle $A_m = (3.14 \pm 0.06)\ \mathrm{cm}^2$		
Volume Ratio of water and bottle	Water (litre)	Final pressure in the rocket p_2 (bar)	Average of u_1 and u_2, $\bar{u}$ (ms^{-1})	Calculated maximum velocity v_{max} (ms^{-1})	Measured maximum velocity (ms^{-1})
-	Error: ±0.005	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$	Error: ± 1.1
$2.0/12 = 0.17$	0.383	2.2	17.6	16.8	20.1
$2.5/12 = 0.21$	0.479	2.1	17.1	18.7	22.5
$3.0/12 = 0.25$	0.575	1.9	16.6	20.0	25.0
$3.5/12 = 0.29$	0.671	1.8	16.0	20.8	24.4
$4.0/12 = 0.33$	0.767	1.6	15.4	21.3	26.0
$4.5/12 = 0.38$	0.863	1.5	14.8	21.4	25.5
$5.0/12 = 0.42$	0.958	1.4	14.0	21.1	25.1
$5.5/12 = 0.46$	1.054	1.2	13.1	20.3	24.1
$6.0/12 = 0.50$	1.150	1.1	12.0	18.6	20.0
$8.0/12 = 0.67$	1.533	0.6 (not exact)	-	-	9.8 (not exact)

Note: The errors were calculated with the same method as in table 3.

The measured and calculated values of v_{max} are also represented in the graph below, with smooth polynomial curves fitted to the data points.

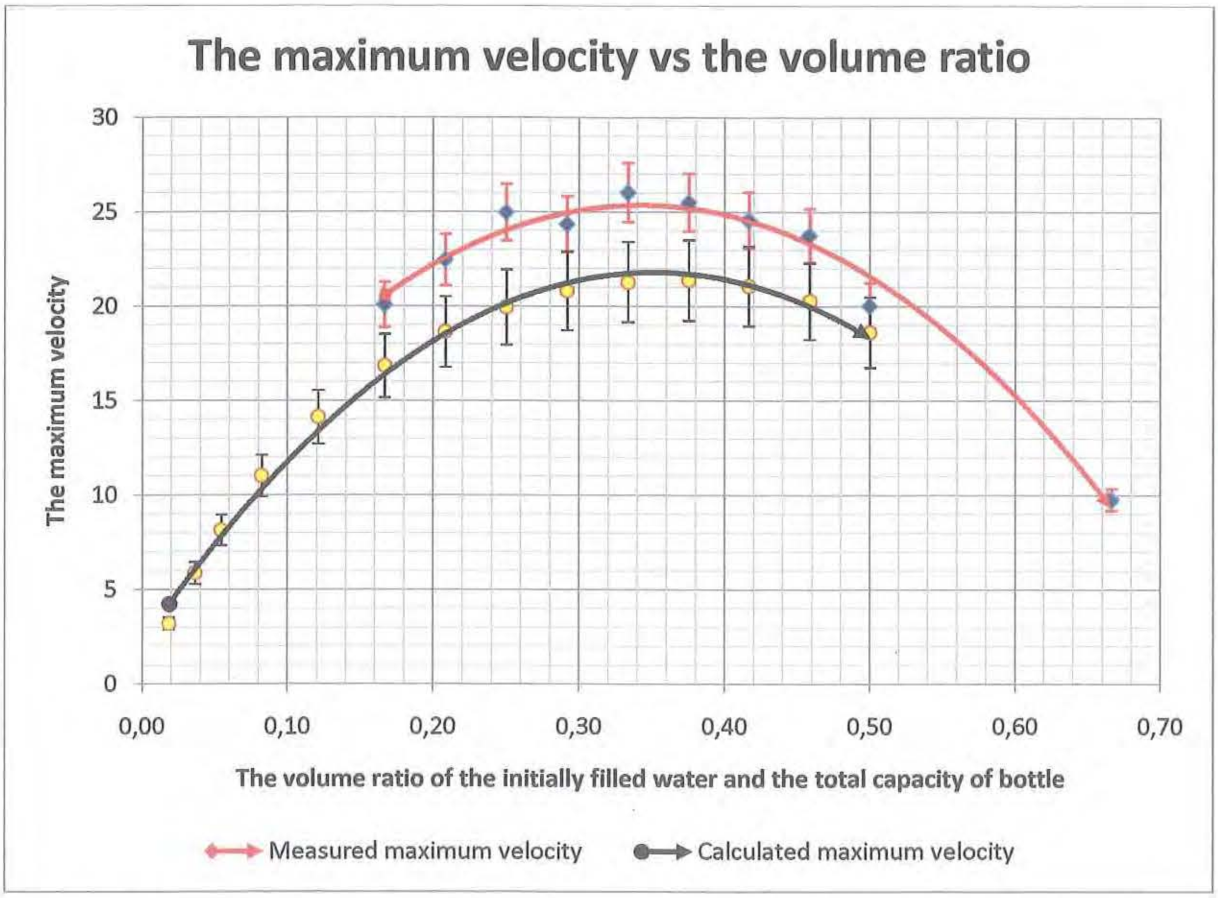


Figure 20: *Shows the calculated and the measured values of the maximum velocity.*

It can be seen that the model's values are smaller by *4 to 5* ms^{-1}, than the measured maximum velocities. The results of the zero litres rocket provide a correction of just the right magnitude (table 5). When the empty bottle was launched at the initial pressure of 1.6 bar it accelerated to 6 to 7 ms^{-1}, which means if the rocket still has a 1.6 bar pressure inside it after the first stage, it can still accelerate despite of the absence of fuel. Therefore, the model can be improved by considering this accelerating force that causes the systematic error.

now it is getting complicated

Table 5: *Shows the experimental result of the zero-litre water rocket*

$0.0/12$	Initial pressure: (1.6±0.2) bar	Water amount: 0 litre	$v_{max} =$ (6.7±0.4) ms^{-1}	$t_t = 0$ s	$h_{max} =$ (2,42±0.02) m

Note: The errors were calculated with the same method as in table 4.

iii. Determining the height reached by the rocket in the 1st phase

by calculation from v(t)

In this experiment, the height h_1 reached by the rocket in the first stage was calculated by integrating (11) from $t = 0\text{s}$ to $t = t_t$, by a graphing software GeoGebra:

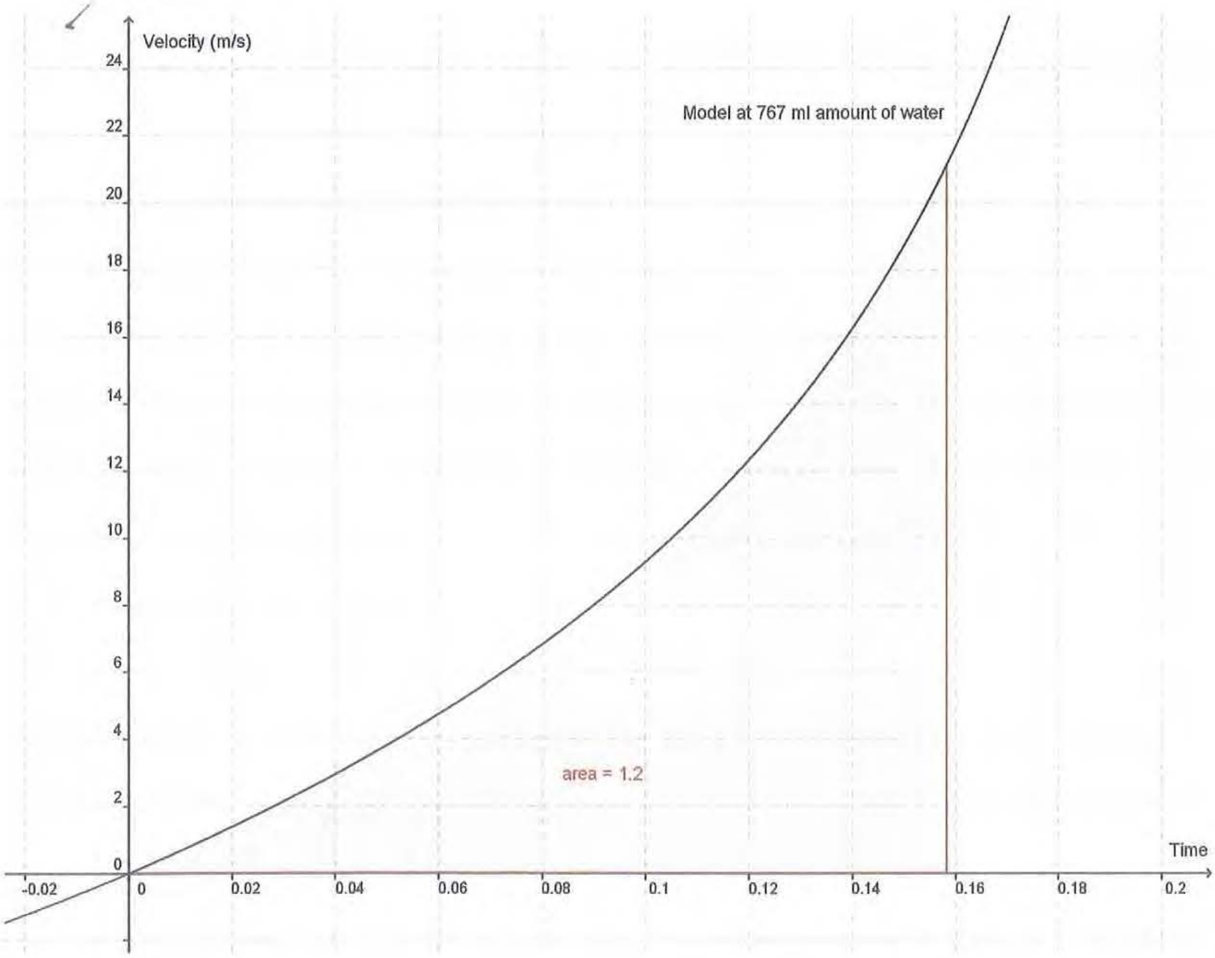


Figure 21: *Shows the method of determining h_1 by integrating (11) from 0 to t_t (shaded in table 6)*

Table 6: *Shows the calculated heights reached by the water rocket in the first phase*

Initial parameters			
Mass of empty rocket $m_R = (0.225 \pm 1)\mathrm{g}$		Density of water: $1000\ \frac{\mathrm{kg}}{\mathrm{m}^3}$	
Average initial pressure p_1: (2.9 ±0.3) bar		Area of nozzle $A_m = (3.14 \pm 0.06)\ \mathrm{cm}^2$	
Amount of water (ml)	Calculated average of $u_1$and u_2 , $\bar{u}$ (ms^{-1})	Calculated time t_t (s)	Calculated height in the 1st phase h_1 (m)
Error: ±5	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$
383	17.6	0.07	0.5
479	17.1	0.09	0.7
575	16.6	0.11	0.9
671	16.0	0.13	1.1
767	15.4	0.16	1.2
863	14.8	0.19	1.4
958	14.0	0.22	1.6
1054	13.1	0.26	1.8
1150	12.0	0.31	1.9
1533	-	-	-

b) The phase when the rocket has run out of fuel

After the ejection of the water ends, the rocket still goes upwards with decreasing velocity. This is the second phase. At the beginning of this stage, the rocket has a mass of m_R and a velocity of v_{max}. Similarly, in this phase the two forces acting oppositely to the direction of the motion of the rocket are gravity $-mg$ and air drag $-Cv$. To obtain the rocket's maximum height, first, the velocity of the rocket in this phase should be expressed.

i. Determining the height reached by the rocket

The motion of the water rocket in the second stage can also be understood by applying Newton's second law: $\sum F = ma$, which in this case is a first order inhomogeneous differential equation [11]: but C is a function of v (not constant)

because of 2nd stage

$$-g = \frac{dv}{dt} + \frac{C_v}{m}v \tag{12}$$

When $t = 0$[8] the velocity of the rocket is v_{max}. Using this initial condition, the velocity in the second phase can be expressed by solving the differential equation:[9]

$$v(t) = \left(v_{\max} + \frac{g}{k}\right)e^{-kt} - \frac{g}{k} \tag{18}$$

Note: For more comfortable notations $\frac{c}{m} = k$. ≠ constant.

After the time of rise t_{fly} to the highest point, the velocity of the rocket is $v = 0$ for it rises until it stops. Using this condition, the time of flight is:

$$t_{fly} = \frac{1}{k}\ln\left[\frac{kv_{\max} + g}{g}\right]$$

To obtain the height reached by the rocket in the second phase h_2, equation (12) should be integrated, and to express the maximum height h_{max} it should be integrated from $t(0)$ to t_{fly} and the height of in the first phase h_1 should be added.

$$h_2 = \int_{t(0)}^{t_{fly}} \left(v_{\max} + \frac{g}{k}\right)e^{-kt} - \frac{g}{k}\, dt$$

$$h_2 = \frac{1}{k}\left[-\left(v_{\max} + \frac{g}{k}\right)e^{-kt_{fly}} - g\ t_{fly} + V_{\max} + \frac{g}{k}\right]$$

error in theory

[8] The time $t=0$ means the beginning of the second stage and the end of the first stage.

[9] The deduction of equation (18) is in the appendix.

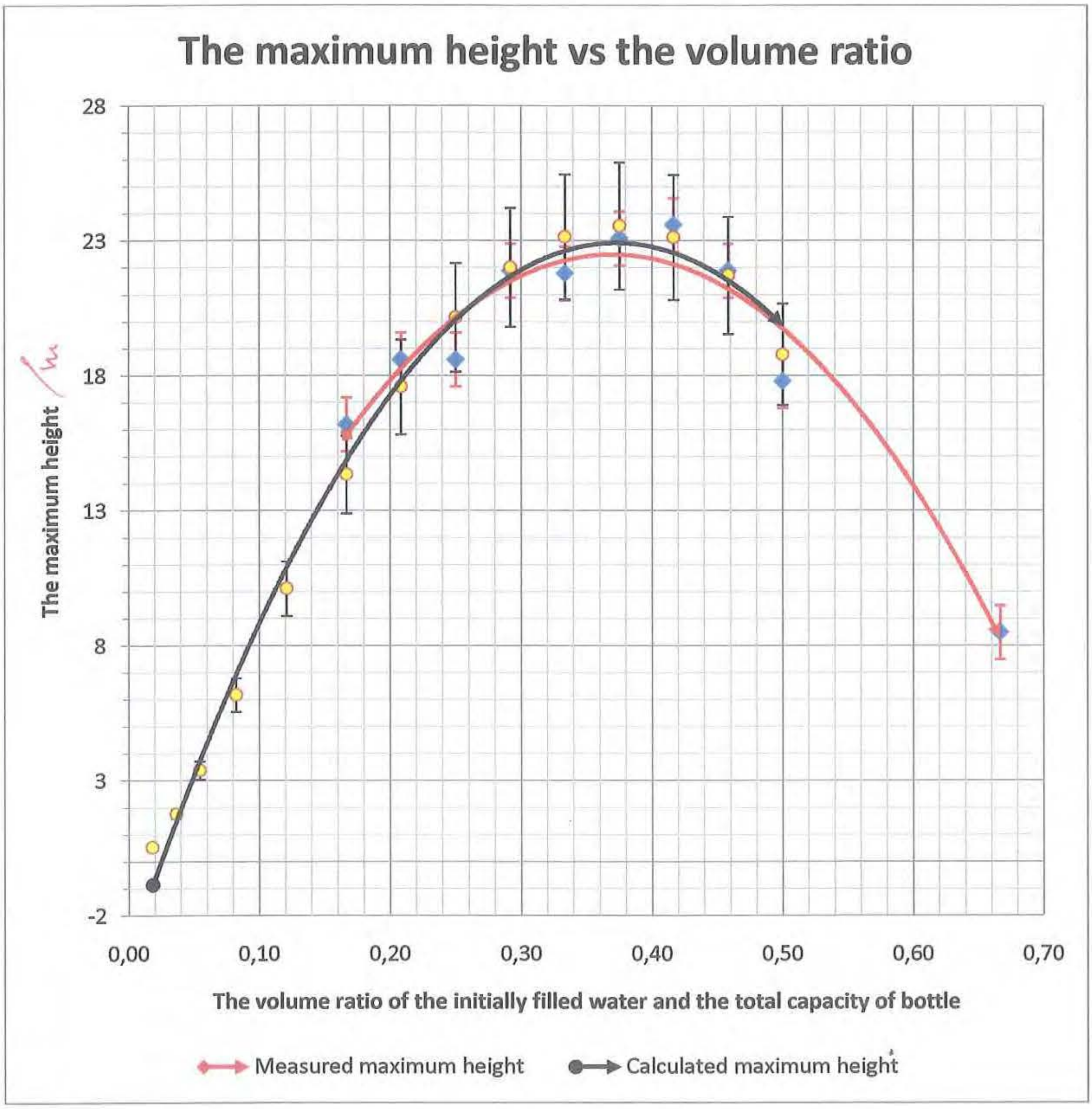
➢ *Calculating the maximum height*

In this section, the same air drag coefficient (*0.75*) is used as in the **Approximation of C** section in the appendix, which means the value of $k = 0.04$.

Initial parameters						
Mass of empty rocket $m_R = (0.225 \pm 1)\text{g}$				The air drag coefficient $c = 0.75$		
Average initial pressure p_1: (2.9 ±0.3) bar				Density of air $\rho_{air} = 1.29 \frac{kg}{m^3}$. [9]		
Largest cross-sectional area $A_L = (80.1 \pm 0.5)\ \text{cm}^2$				Value of $k = 0.04$		
Water and bottle volume ratio	V_0 (litre)	Calculated from model				Measured h_{max} (m) from table 2
		t_{fly}(s)	Height in the 1^{st} phase h_1 (m) from table 6	Height in the 2^{nd} phase h_2 (m)	Calculated maximum height $h_{max} = h_1 + h_2$ (m)	
	Error: ±0.005	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$	Error: $\pm\approx 10\%$	Error:±0.5
0.17	0,383	1.66	0.5	13.9	14.4	16.2
0.21	0,479	1.84	0.7	16.9	17.6	18.6
0.25	0,575	1.96	0.9	19.3	20.2	18.6
0.29	0,671	2.04	1.1	21.0	22.1	21.9
0.33	0,767	2.09	1.2	21.9	23.1	21.8
0.38	0,863	2.1	1.4	22.1	23.5	23.1
0.42	0,958	2.07	1.6	21.5	23.1	23.6
0.46	1,054	1.99	1.8	19.9	21.7	21.9
0.50	1,150	1.83	1.9	16.9	18.8	17.8
0.67	1,533	-	-	-	-	8.5

Table 7: *Shows the variables needed to determine the maximum height that the water rocket can reach*

Figure 22: *Shows the calculated and the measured values of the maximum height reached by the water rocket at the initial pressure of 2.9 bar*



The heights calculated from the model fit the measured heights very well. The maximum of each curve, as read from the graphs, occurs at 0.37 ± 0.02; which is the optimal volume ratio of the water and bottle.

wrong, since $\Delta p \sim 10\%$ for a start

IV. CONCLUSION

larger uncertain

In spite of the large uncertainty in the initial pressure, the model is able to predict the optimal volume ratio of water and bottle 0.37 ± 0.02, which is also obtained by experiment.

However, the hypothesized value of $\frac{1}{3} = 0.33$ and $40\% = 0.40$ to $50\% = 0.50$ does not lie within the interval 0.37 ± 0.02, therefore, the statements that the optimal value is $\frac{1}{3}$ or between 40% and 50% of the volume of the bottle are not justified by the experiment. It is only acceptable as a rough estimation.

The applicability of the model is limited to the case when all the water leaves the bottle by the end of the first phase of the movement of the rocket. In addition, the model can be refined by considering the effect of the acceleration caused by the air coming out or the variation of the mass by changing the weight on the nose of the rocket.

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Only final ± ~~need~~ disqualifies from 100%!

excellent in all respects. A few technical glitches, but candidate was using sophisticated maths – so these can be forgiven. Esp. good to see a model being explained well.

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Accessed: 31/10/2011

VI. Acknowledgements

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VII. Appendix

a) The unrounded values of the experimental results produced by Logger Pro:

The average initial pressure p_1: (2.9 ± 0.3) bar				
The ratio	Initial water amount V_0 (ml)	Maximum velocity v_{max} (ms^{-1})	Time of ejection of water t_t (s)	Maximum height h_{max} (m)
	error: $\pm$5ml	error: $\pm 1.14381\ \mathrm{ms}^{-1}$	error: ±0.0034	error: $\pm$0.5m
$^{2.0}/_{12}$	383	21,08582	0,0704	16.2
$^{2.5}/_{12}$	479	23.96376	0,0876	18.6
$^{3.0}/_{12}$	575	25.98061	0,0875	19.7
$^{3.5}/_{12}$	671	25.35379	0,1418	21.9
$^{4.0}/_{12}$	767	26.03682	0,1293	21.8
$^{4.5}/_{12}$	863	25.84454	0.1668	22.1
$^{5.0}/_{12}$	958	26.57256	0,1793	23.6
$^{5.5}/_{12}$	1054	24.76175	0,2023	21.9
$^{6.0}/_{12}$	1150	21.04051	0,2544	16.8
$^{8.0}/_{12}$	1533	10.77093	-	8.5
$^{0.0}/_{12}$	initial pressure: (1.6±0.2) bar water amount: 0*l*	(6.743±0.3947) ms^{-1}	0 s	(2.421 ± 0.024) m

Table 8: *Shows the measured values of (table 2 and table 5)*

b) The approximation of C:

$-Cv(t) - m(t)g = \frac{dv}{dt}m(t) + \frac{dm}{dt}\bar{u}$, where *C* depends on the density of air $\rho_{air} = 1.29\frac{kg}{m^3}$, the air drag coefficient *c*, and the largest cross-section of the rocket A_L.

Dividing the equation by $m(t)$and rearranging it:

$$\frac{dv}{dt} = -\frac{dm}{m(t)}\frac{\bar{u}}{dt} - g - \frac{Cv(t)}{m(t)}$$

This equation can only be solved numerically because both the mass and the velocity are changing with time, so the term of the air drag should be approximated to see how much it would vary in the maximum velocity.

Note: Since the rocket does not go really fast, an assumption can be made that the force of air drag is proportional to its velocity [9].

$$dv \approx \int \frac{Cv\,dt}{m(t)} \approx \int \frac{Cv\,dt}{(V_0{-}\bar{u}A_m t)\rho_w + m_R}$$

To solve the integral an assumption should be made that the mass of the rocket is not changing, so an arithmetic mean of the initial and the final mass should be used.

According to several articles and scientific investigations [1, 5, 6] the air drag coefficient can range from *0.2 to 0.75.* To see how much the air drag maximum influences the rocket's velocity in the first phase, the largest value for the velocity should be expressed:

$$C = 1.29 \times 0.75 \times 0.00801 = 0.01$$

The arithmetic mean of the smallest mass (at $\frac{2.0}{12}$) is:

$$\frac{(V\rho_{w_{0-}} + m_R) + m_R}{2} = \frac{1}{2}V_0\rho_w + m_R = 0.192 + 0.225 = 0.417$$

Since C and the arithmetic mean of the mass are constants they can be brought out from the integral and the integral of the velocity is the height it reached in the first phase, which, according to the videos is at most 2 meters:

$$\frac{0.01\int v\,dt}{0.417} = \frac{0.01 \times 2}{0.417} = 0.048\ \mathrm{ms}^{-1}$$

This term of the equation can be neglected for it is much smaller compared to the measured maximum velocity of $16.8\mathrm{ms}^{-1}$ at $\frac{2.0}{12}$ from table 4 (shaded).

c) The deduction of equation (7):

$$P(t) = v(t)m(t) \tag{4}$$

$$p(t+\Delta t) = (v(t)+\Delta v)(m(t)+\Delta m), \tag{5}$$

$$p_w(t+\Delta t) = -(\bar{u} - v(t)) \times |\Delta m| = \Delta m(\bar{u} - v(t)) \tag{6}$$

If equations (4;5;6) are substituted into Newton's second law the equation looks like this:

$$F = \lim_{\Delta t \to 0} \left[\frac{(v(t) + \Delta v)(m(t) + \Delta m) + \Delta m(u(t) - v(t))}{\Delta t} \right]$$

$$F = \lim_{\Delta t \to 0} \left[\frac{v(t)m(t) + v(t)\Delta m + m(t)\Delta v + \Delta v \Delta m + \Delta m u(t) - v(t)\Delta m}{\Delta t} \right]$$

After the algebraic simplifications the equation looks like the following:

$$F = \lim_{\Delta t \to 0} \left(\frac{m(t)\Delta v + \Delta m \Delta v + \Delta m \bar{u}}{\Delta t} \right) \tag{7}$$

d) The deduction of equation (11)

$$-m(t)g = \frac{dv}{dt} m(t) + \frac{dm}{dt} \bar{u} \tag{8}$$

After the division and the rewriting of the mass that it equals the volume of the water times the density of the water, the rearranged equation should look like this:

$$\frac{dv}{dt} = -\frac{\frac{dV}{dt}\rho_w}{V(t)\rho_w + m_R} \times \bar{u} - g \tag{9}$$

From the assumption before, the rate of decrease of the volume of the rocket is:

$$-\frac{dV}{dt} = \bar{u}A_k$$

So, the rocket's volume at time *t is*:

$$V(t) = V_0 - \bar{u}A_k t$$

Substituting the volumes back to equation (9):

$$dv = \frac{\bar{u}^2 A_k \rho_w \, dt}{(V_0 - \bar{u}A_k t)\rho_w + m_R} - g \, dt \tag{10}$$

Let the denominator, which is the mass of the rocket at time moment *t*, be y:

$$y = (V_0 - \bar{u}A_k t)\rho_w + m_R$$

Expressing *t* to determine *dt* to substitute back to equation (10):

$$-\frac{y - m_R}{\bar{u}A_k \rho_w} - V_0 = t$$

$$dt = -\frac{dy}{\bar{u}A_k \rho_w} \rightarrow dv = -\frac{dy}{y}\bar{u} - g dt$$

After the definite integration of the equation from t_0 to t and the substitution of y and y_0 back to the equation, the velocity at any time t can be expressed:

$$v(t) = v_0 + \bar{u}\, ln\left[\frac{V_0\rho_w + m_R}{(V`_0 - \bar{u}A_k t)\rho_w + m_R}\right] - g(t) \tag{11}$$

e) The deduction of equation (18)

The motion of the water rocket in the second stage applies Newton's second law: $\sum F = ma$, which in this case is a first order inhomogeneous differential equation:

$$-g = \frac{dv}{dt} + \frac{C}{m}v \tag{12}$$

Note: The initial condition for this differential equation is when $t = 0$ (the time *t=0* means the beginning of the second stage and the end of the first stage) the velocity of the rocket is v_{max}.

To solve an inhomogeneous differential equation, first the homogeneous equation is solved and then add to a particular solution of the equation [11] which in this case is: $v_{inhom} = v_{hom} + v_P$

The homogeneous differential equation:

$$0 = \frac{dv}{dt} + kv\,, \tag{13}$$

where $k = \frac{c}{m}$.

The solution of (13) has to be sought in the form:

$v_{hom} = Ae^{\lambda t}$, where A and λ is a constant and t is time. Differentiating this equation will give:

$$\frac{dv_{hom}}{dt} = A\lambda e^{\lambda t}$$

Substituting back to the homogeneous differential equation (13):

$$0 = A\lambda e^{\lambda t} + kAe^{\lambda t} \tag{14}$$

Simplifying this equation: $0 = \lambda + k$, which follows:

$$\lambda = -k \tag{15}$$

If (15) is substituted back to the equation (14), the homogeneous velocity should be expressed as:

$$v_{hom} = Ae^{\lambda t} = Ae^{-kt} \tag{16}$$

Now, a particular solution should be chosen:

If $\frac{dv_p}{dt}$ is chosen to be 0 then from the equation (12), $\frac{dv_p}{dt} + v_p k = -g$, $v_p = -\frac{g}{k}$ can be expressed so: $0 + \left(-\frac{g}{k}\right)k = -g$

This follows that, $v_{inhom} = v_{hom} + v_P = Ae^{-kt} + \left(-\frac{g}{k}\right)$

which means:

$$v(t) = Ae^{-kt} - \frac{g}{k} \qquad (17)$$

Using the initial condition: the velocity is at maximum when $t = 0$, to solve equation (17)

$$v_{max} = A \times 1 - \frac{g}{k}$$

$$A = v_{max} + \frac{g}{k}$$

Substituting the value of *A* back to equation (17) will give the velocity in the first phase at any time *t:*

$$v(t) = \left(v_{max} + \frac{g}{k}\right)e^{-kt} - \frac{g}{k} \qquad (18)$$

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