Mathematics Extended Essay:

How can one predict what a quiggle will look like without drawing it?

Name: Alex Hall

Centre: Waterford Kamhlaba UWCSA

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Word count: 3938

Abstract

In my essay, I analysed the patterns produced by a program I created. I discovered the patterns myself, and named them quiggles. A quiggle is formed when a point takes a series of steps in a certain way and traces its path out in the process. The point must turn every step, and the amount by which it turns must increase by a constant every step.

The first thing that I noticed about quiggles was that when the aforementioned constant is changed, the quiggle changes drastically and unpredictably. My research question was therefore:

"How can one predict what a quiggle will look like without drawing it?"

To answer this question, the essay first introduces some notation, the most important being θ (the direction in which the point moves), t (the angle by which the point turns) and *a* (the constant by which t increases each step).

Quiggles are then classified into three basic types by observing the program: z-quiggles, o-quiggles, and straight quiggles. Each type is defined by the number of features, which are congruent shapes in a quiggle.

After analysing the formation of z-quiggles, I derive the formula $n = \frac{360k}{a}$ for the number of steps n required to form a feature. This is then used in the next formula that I derive: $\theta'_{n+1} = \frac{n}{2}[2t_1 + (n-1)a]$. This gives the angle of rotation between features in a quiggle. If it is 0, the quiggle is straight, or a hypothetical mono-quiggle.

The last derived formula $p = \frac{360}{x} r$ gives the number of features p: if p=2 then it is a z-quiggle, if p>2 then it is an o-quiggle, whose precise shape depends on the value of r.

The essay concludes with the full process of prediction using these formulae and some other questions concerning quiggles that are worth investigating.

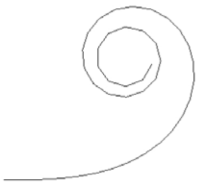
Word Count: 290

Contents

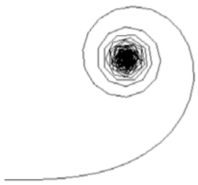
Introduction

Imagine a point in 2-dimensional space. The point takes a step forward in a certain direction. It then turns by an angle of 1°, and takes another step. Then it turns by an angle of 2°, and takes a third step. Continuing in this manner, increasing the 'turning angle' by 1° with each step (all of equal distance), the point traces out its path, producing a beautiful curve. The formation of this curve looks like this:

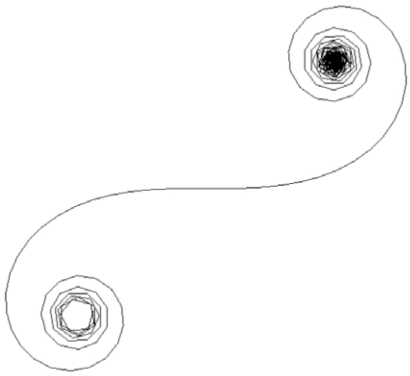
After 40 steps:



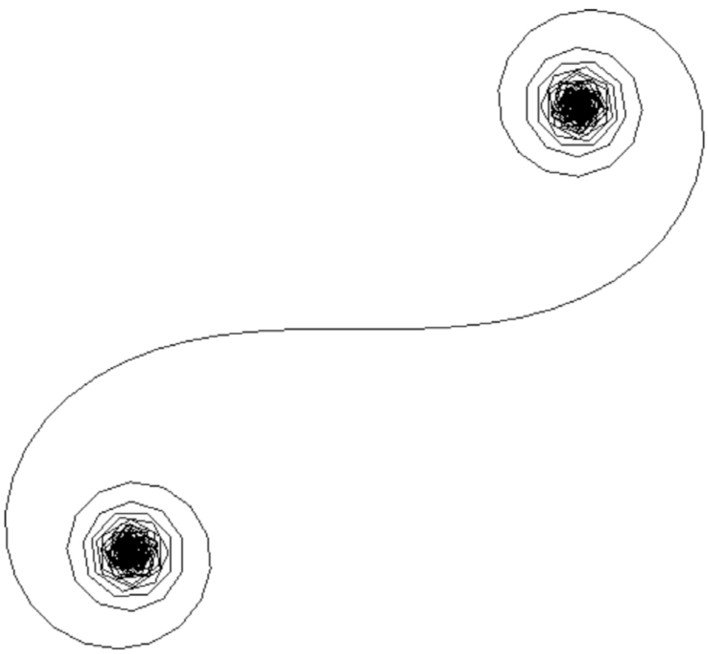
After 180 steps:



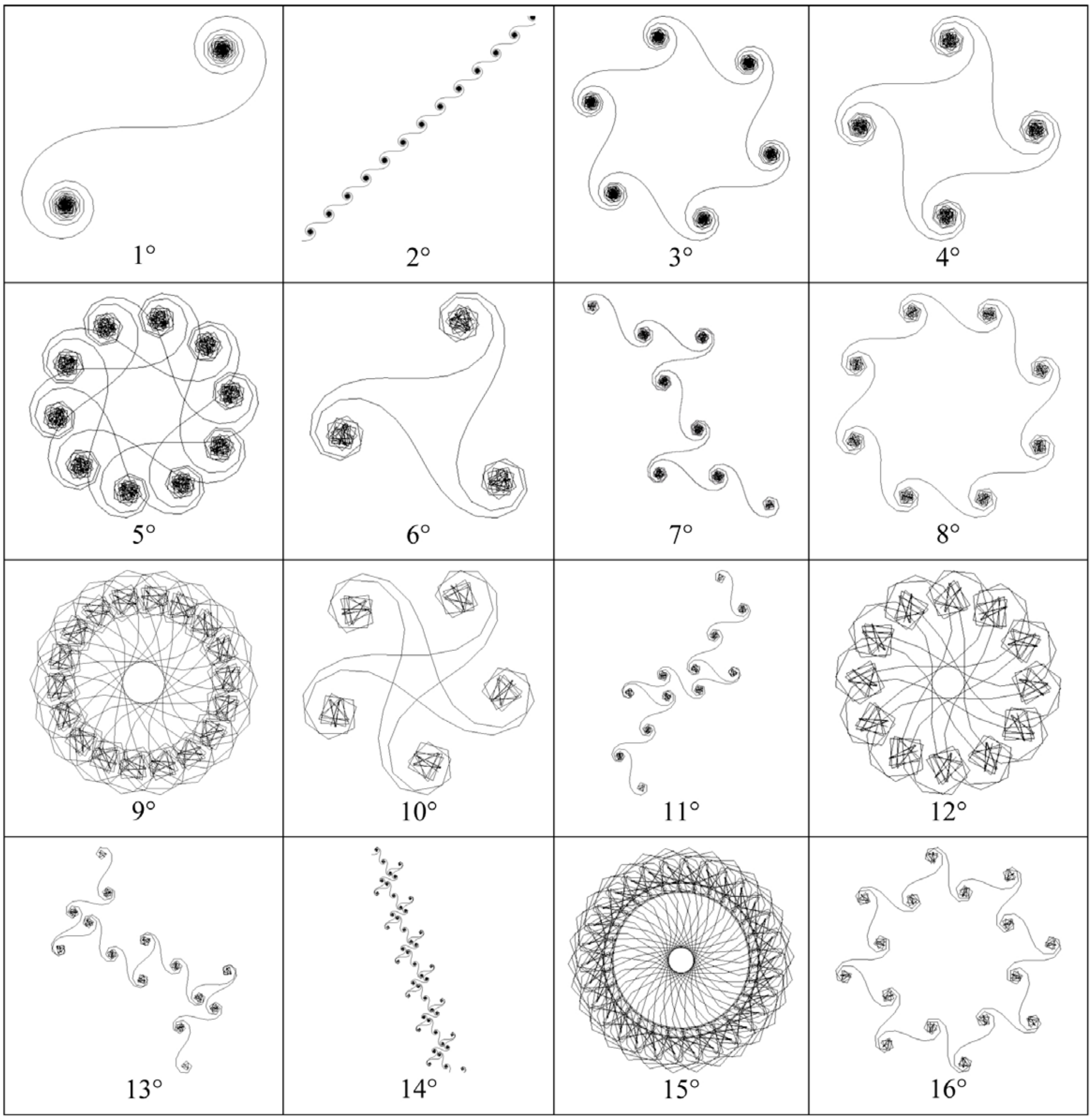
After 430 steps:



After 720 steps, the point has returned to its original starting position, and from there it exactly retraces its steps indefinitely. The curve is complete:



I wrote a simple computer program to create this curve. If the program is changed to increase the turning angle by 2° each step (such that the point turns by 1°, then 3°, then 5°, 7°, etc.) the result is surprisingly different. The curve continues to trace the same basic pattern again and again, stretching out into infinity. Changing the increase to 3° gives an even more drastically different result. Clearly, the value of this increase is a crucial parameter in determining the curve's appearance. The table below shows all the patterns produced when the parameter has any integer value from 1° to 16°. They are not all drawn to the same scale.



I could not find anything on the Internet about these shapes – they are an independent discovery. Therefore I have invented a name for them: quiggles.

The table on the previous page shows that quiggles appear to be highly unpredictable. Each shape almost seems random with respect to the parameter, and curiosity drives me to investigate them and show that they are not random by answering the question:

How can one predict what a quiggle will look like without drawing it?

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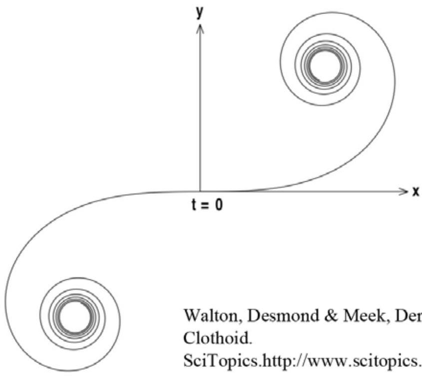
Possible applications

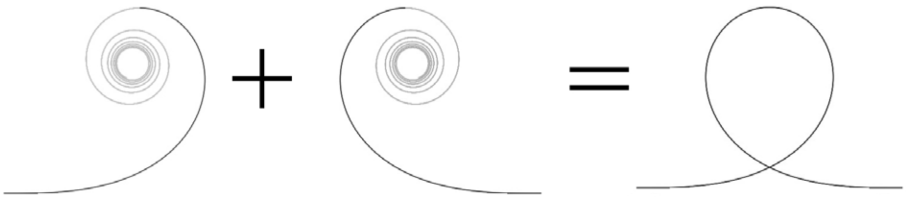
I found one thing that looks similar to quiggles (picture retrieved from website [2]):



Walton, Desmond & Meek, Dereck (2008, October 7). The Euler/Cornu spiral, Clothoid. SciTopics.http://www.scitopics.com/The_Euler_Cornu_spiral_Clothoid.html

This has different names: an Euler spiral, a Cornu spiral, or a clothoid. It has linearly increasing curvature, just as a quiggle forms from a point that gradually turns more and more. The main difference between quiggles and clothoids is that quiggles are produced by discrete steps, whereas clothoids are produced by parametric equations [2], and are perfectly smooth, continuous curves.

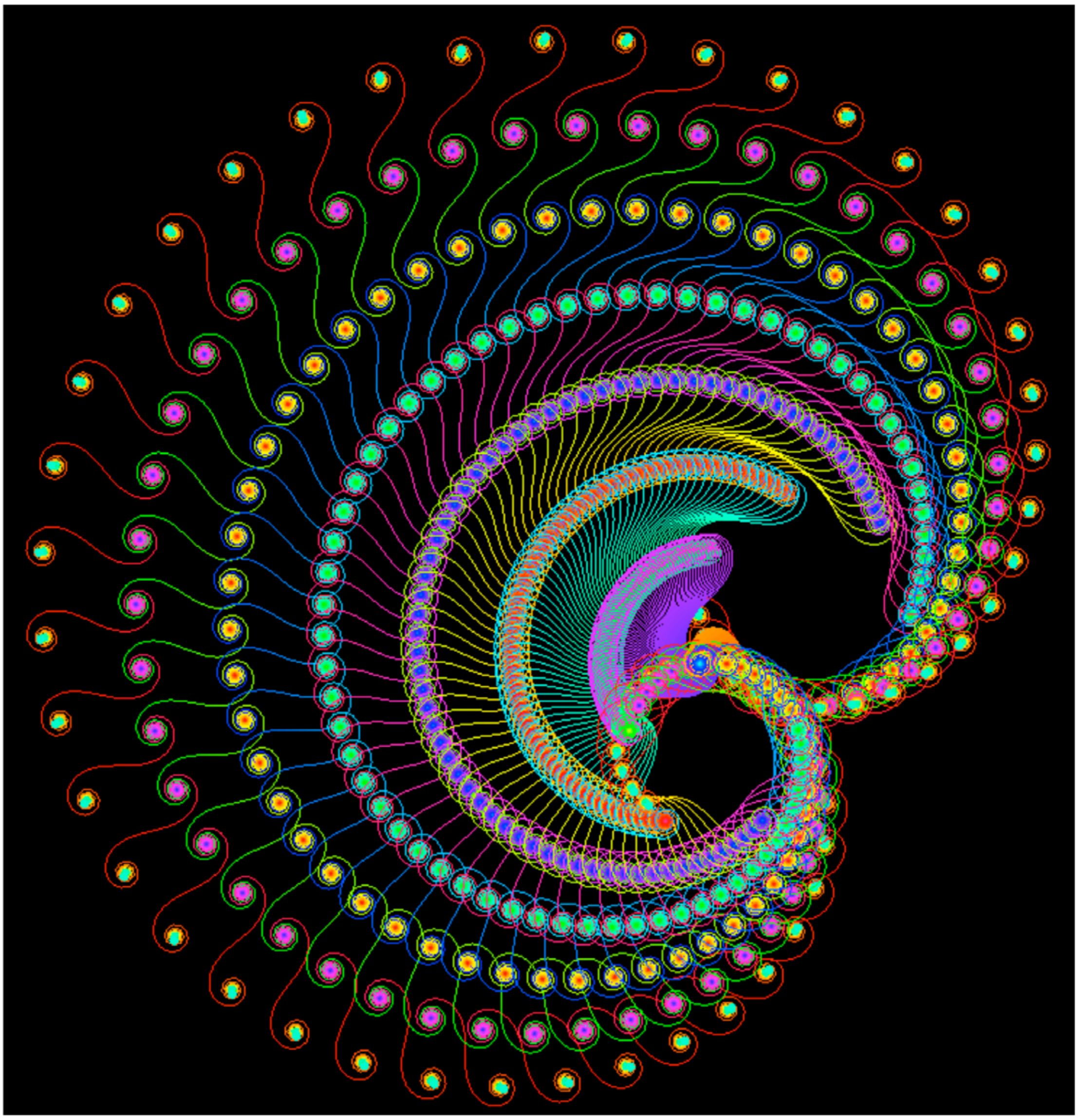
Clothoids are not just pretty spirals. For example, they are related to Fresnel integrals, which are important in studying the diffraction of light [2]. Another important characteristic of a clothoid is its varying radius of curvature, so sections of it are used in roller coaster tracks to create the maximum acceleration at the peak of a loop [1]:



$=$

(picture from website [1])

This means that quiggles might have some real life use. Of course, they are also beautiful. The image below is produced by superimposing several incomplete quiggles which all have a slightly different value of t_1, a parameter which is about to be introduced.



Notation

To describe quiggles and their formation and answer the research question, some formal notation must be introduced, especially the variables involved when a quiggle is drawn.

Two values constantly change throughout the formation of a quiggle. The first is the direction in which the point is moving, since after every step the point turns. The second is the turning angle: the angle by which the point turns every step, or the value by which the direction changes every step (remember that the point turns a different amount every step). Both of these values are angles measured in degrees, and henceforth they always represent a number of degrees. If we treat them as angles, then that means that when they are greater than 360, we simply subtract 360 till they are less than 360 again – e.g. 810 is the same as 450, which is the same as 90. When dealing with calculations, though, it is usually useful at first not to truncate the value to less than 360. Therefore we have two ways of denoting each value:

θ' : the 'full' value of the direction in which the point takes a step.
t' : the full value of the turning angle. Every step, θ' is increased by this value.
θ : the remainder of θ' modulo 360, i.e. the direction treated as an angle between 0 and 360. Specifically, θ is the angle between the direction of the point and the horizontal, measured anticlockwise.
t : similarly, this is t' treated as an angle. t stands for 'turn'. If t is positive, then the point turns anticlockwise, since θ is measured anticlockwise.

Remember that these values change every step. In some contexts, it is useful to simply write that "θ reaches" or "t' increases to" or a similar phrase, whereas sometimes it is preferable to talk about the value of these variables at specific times during the quiggle's formation. In these cases, I will write θ_n or t_n to denote the value at the n^{th} step, e.g. in the 5^{th} step the point moves in the direction θ_5.

a : stands for 'angular acceleration'. Every step, t' is increased by a. This value is constant and setting a greater than 360 would be redundant, so there is no need to denote anything by a_n or a'.

This can be summarised by the following relations:

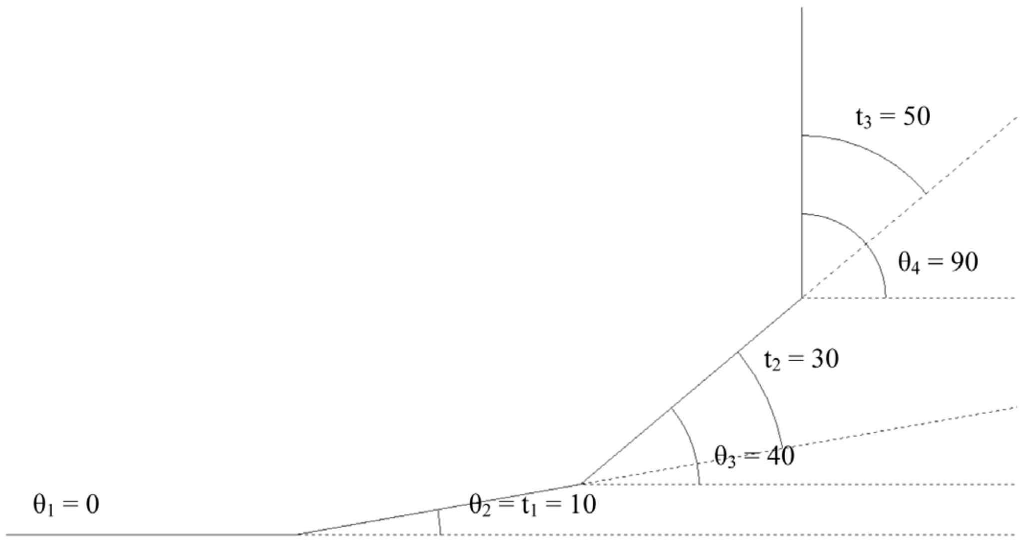
$\theta'_{n+1} = \theta'_n + t'_n$ $\qquad$ $\theta' \equiv \theta \pmod{360},\ \theta < 360$

$t'_{n+1} = t'_n + a$ $\qquad$ $t' \equiv t \pmod{360},\ t < 360$

From these relations, the value of θ_n can be calculated for all natural values of n (and hence the full quiggle can be generated by moving the point in the direction θ_n in the n^{th} step) if we are given the values of θ'_1, t'_1, and a to start with. We will always set θ'_1 to 0, because increasing it would increase every other value of θ_n by the exact same amount, which would only rotate the quiggle. t'_1 and a are set within the program before generating a quiggle, and they are the main parameters that are important in the formation of the quiggle. Also, t'_1 and t_1 are equivalent, since setting t'_1 higher than 360 would be pointless.

An alternate, unofficial, more precise formulation of the research question could therefore be: "Based on the parameters a and t_1, how can one predict what a quiggle will look like without drawing it?"

To clarify these variables and the process of forming a quiggle, the diagram below shows the first 4 steps of an example quiggle where $a = 20$ and $t_1 = 10$. Remember that the point turns anticlockwise and that θ is measured as an angle with the horizontal.



It is also important to note for future reference that the values t'_1, t'_2, t'_3… form an arithmetic progression with common difference a and first term t_1. From this we can deduce that:

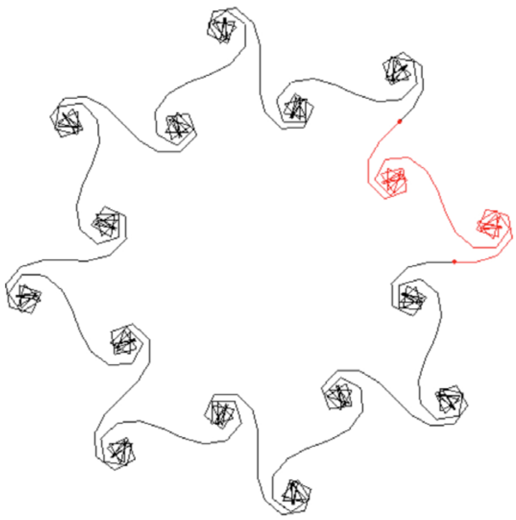
$$t'_{n+1} = t'_1 + an$$

We can also see that θ'_{n+1} is the sum of the first n terms of this progression.

Finally, a quiggle where $t_1 = 1$ and a is some integer b is called a b-quiggle, substituting b with its value if it is known. These are easier to work with but really a and t_1 can be any rational numbers.

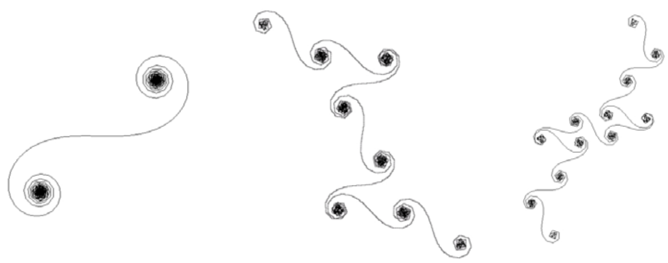
Classification

The next step to understanding quiggles is to classify them. This was done simply by observing quiggles generated by the program, not by mathematical reasoning. First, the concept of 'features' must be explained. A feature is defined as a shape that is repeatedly drawn in a quiggle, where each feature is congruent. For example, the 16-quiggle below has 8 features, one of which is highlighted in red, with its endpoints marked.



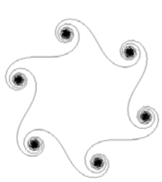
There are three basic types of quiggle, each defined by the number of features it has:

z-quiggles have 2 features attached to the centre, each a 180° rotation of the other. Some examples are:

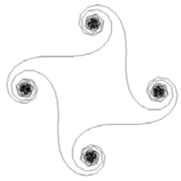


1-quiggle 7-quiggle 11-quiggle

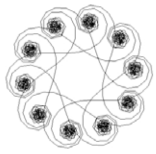
o-quiggles have more than 2 features, with rotational symmetry equal to the number of features:



3-quiggle



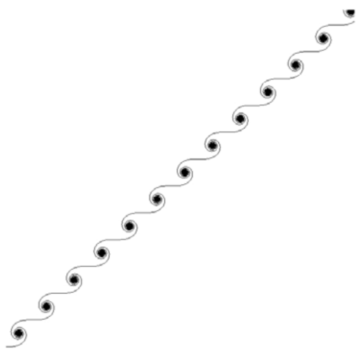
4-quiggle



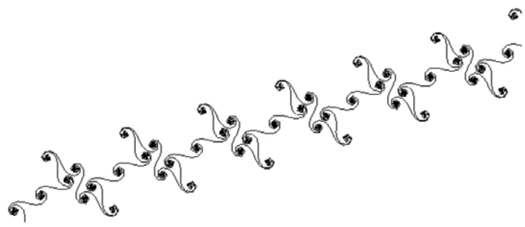
5-quiggle

In a sense, z-quiggles are a special case of o-quiggles, although I prefer to consider them separate, just as a line segment is not a polygon.

Straight quiggles continue indefinitely in the same direction, so they have an infinite number of features, giving them translational symmetry:



2-quiggle



14-quiggle

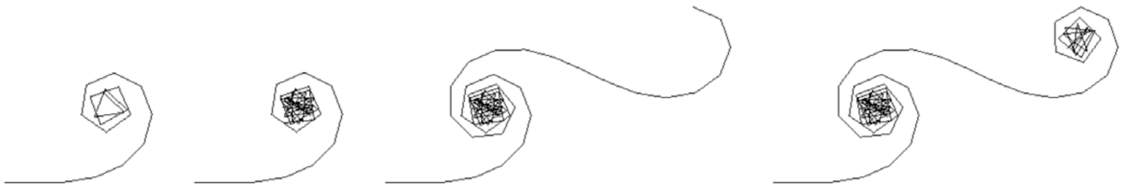
The table below shows what values of b correspond to what shapes for all b-quiggles. Refer to the table of quiggles on page 2 for comparison.

a	Shape	*a*	Shape	*a*	Shape	*a*	Shape	*a*	Shape	*a*	Shape
1	z	6	o	11	z	16	o	21	o	26	Straight
2	Straight	7	z	12	o	17	z	22	Straight	27	o
3	o	8	o	13	z	18	o	23	z	28	o
4	o	9	o	14	Straight	19	z	24	o	29	z
5	o	10	o	15	o	20	o	25	o	30	o

By inspecting these values, a crude method for predicting the shape of a quiggle can be conjectured. If *a* and 360 are relatively prime, a z-quiggle is produced. If the greatest common divisor (GCD) of *a* and 360 is 2, a straight quiggle is produced. All the remaining quiggles will have a GCD between a and 360 that is greater than 2, and these are o-quiggles. This method only works if a is an integer and $t_1 = 1$, but it's a good place to start when trying to explain how quiggles work. The conjecture that this method works will be proved in the following sections.

The formation of z-quiggles

Something strange happens when a z-quiggle is formed. For example, take the 7-quiggle. At first, lines are drawn normally.



20 steps 40 steps 60 steps 78 steps

After 78 steps, though, everything freezes temporarily. It seems that nothing new is being drawn, until suddenly new lines emerge from the other end after exactly another 78 steps.

After 170 steps

What happened? We can see by calculating t_{78}:

$$t'_{n+1} = t'_1 + an$$

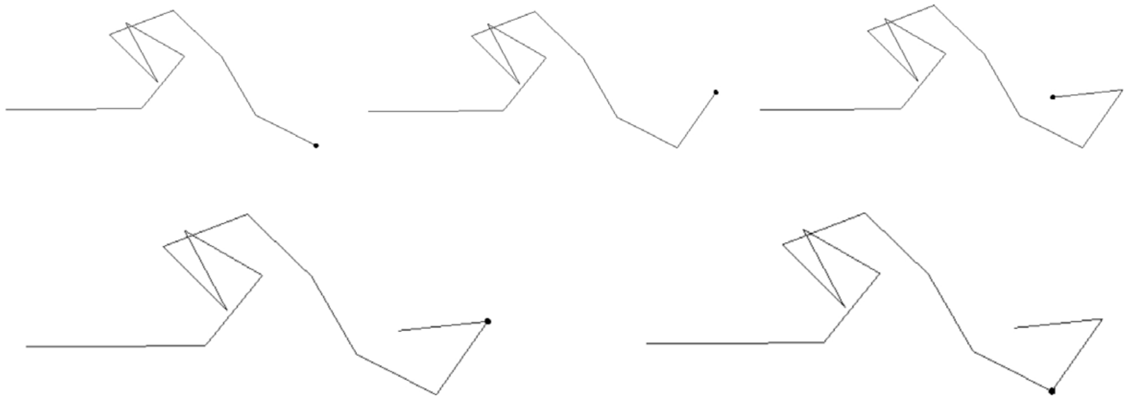
$$\Rightarrow \quad t'_{78} = 1 + 7 \times 77$$

$$= 540$$

$$\Rightarrow \quad t_{78} = 180$$

So after the 78th step, the point turned around 180° and retraced its last step. All the steps after that also retraced previous steps, (we shall soon see why) as though the point was walking backwards, until it reached the starting point.

Observing the program suggests that all z-quiggles do this. The critical step of turning 180° is shown here for the 49-quiggle. The position of the point is highlighted by a small circle.



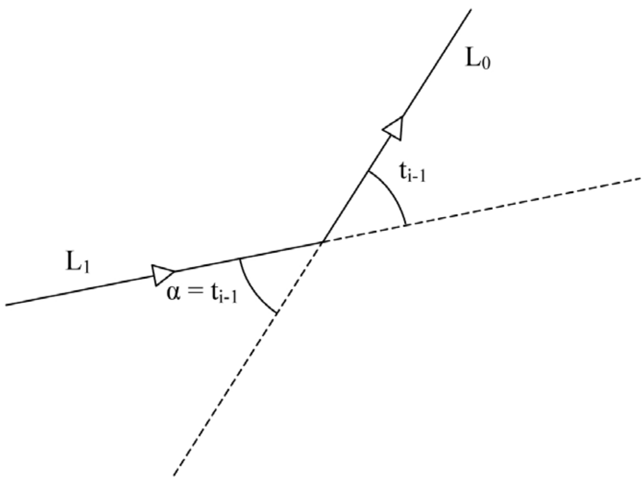
After the point reaches its original location again, it starts to trace out new line segments, until eventually t = 180 again. Then the point will retrace its steps again, and no new steps will be formed. This will leave two features, each a 180° rotation of the other (this fact will be proven later). For now, two facts need proving:

1. If at any step t = 180, then all the following steps will retrace any steps that have already been made.
2. For any b-quiggle, t will eventually be 180 (so a z-quiggle will be formed) if and only if *a* and 360 are relatively prime, as conjectured in the classification section.

To prove the first fact, look at what happens when the point moves in the steps directly before and after turning 180°. If we let $t_i = 180$, then at some step the point will do the following:

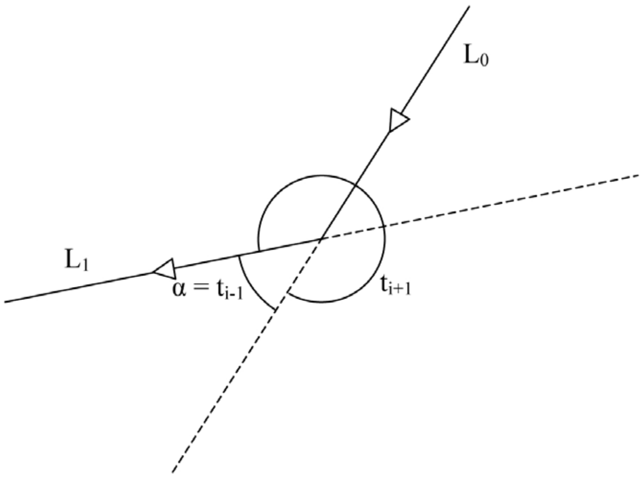
1. Draw a line (call it L_1).
2. Turn by t_{i-1}.
3. Draw another line (call it L_0).
4. Turn by $t_i = 180$.
5. Retrace L_0.
6. Turn by t_{i+1}.
7. Draw another line segment.

We need to prove that the line segment in the final step is the same as L_1, i.e. L_1 was retraced. The first 3 steps are shown in this diagram:



Dotted lines represent extensions of the point's path to show how it turns, and arrows show the direction in which the point moves. Remember that the point always turns anticlockwise. Also note that the angle marked α is not an angle turned, but is equal to t_{i-1} because of vertex angles.

If L_1 is retraced, then the following situation occurs when the point returns (steps 5 to 7):



Again, it's important to remember that the point always turns anticlockwise to see why that particular angle is chosen as t_{i+1}. It's clear from the diagram that if L_1 is indeed retraced, then:

$$t_{i-1} + t_{i+1} = 360$$

We can see that this result is true by looking, for example, at successive values of t when $a = 7$:

…152, 159, 166, 173, **180**, 187, 194, 201, 208…

Starting from 180 in the middle, if you pick a number a few jumps to the left and another number the same number of jumps to the right, they will add up to 360. That is, if j is some integer, then:

$$t_{i-j} + t_{i+j} = 360^{*}$$

For example, if $j = 2$, then $t_{i-j} + t_{i+j} = 166 + 194 = 360$.

The case in the diagrams above, when $j = 1$, is only one case where all this applies. The numerical and geometric arguments just used can apply to all retraced steps. They serve as an inductive step, proving that if one step is retraced, then the next step will be retraced as well (unless there are none left to retrace). We also have an initial case, because if t is ever 180, then the point will surely

* This is considered to be a simple, standard result of arithmetic progressions, so see appendix for proof

retrace one step. Therefore we have a complete proof by induction that if, at any time, t = 180, then all the following steps will retrace any steps that have already been made, thus forming a z-quiggle. We also need to prove that for a b-quiggle, t will at some step equal 180 *if and only if* *a* and 360 are relatively prime. Proving this one way is easy. Let *a* and 360 be coprime. Since t′ increases by *a* every step, then its remainder when divided by 360 (i.e. t) will cycle through all integers from 0 to 359 (this is a standard result in number theory), including 180.

The inverse of this statement is that if *a* is an integer, $t_1 = 1$, and *a* and 360 have a GCD greater than 1 (meaning they are not coprime), then t will never be 180.

Henceforth, let the GCD of *a* and 360 be d.

It's been given that:

$$t'_{n+1} = t'_n + a$$

and

$$t' \equiv t \pmod{360},\ t < 360$$

$$\therefore\ t_{n+1} \equiv t_n + a \pmod{360}$$

Since 360 and *a* are multiples of d, it follows that:

$$t_{n+1} \equiv t_n + a \pmod{d}$$

$$\therefore\ t_{n+1} \equiv t_n \pmod{d}$$

Since $t_1 = 1$, this proves by induction that:

$$t_n \equiv 1 \pmod{d}\ \forall\ n \in \mathbb{N}$$

This means that t is never a multiple of d, and since 180 is a multiple of d, this shows that t is never 180. Therefore we have proven the inverse, and overall we have proven that:

$$\text{If } a \in \mathbb{Z} \text{ and } t_1 = 1 \text{ then } (\exists\ i \in \mathbb{N} \text{ such that } t_i = 180 \Leftrightarrow \mathrm{GCD}(a, 360) = 1)$$

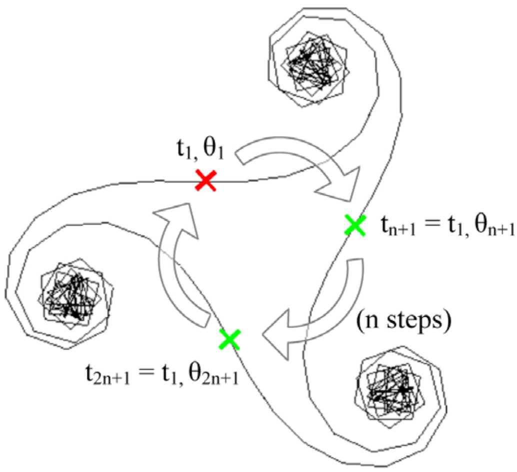
The rotation of features

To predict what a quiggle will look like, its features must be analysed. Firstly, how are features formed? What must happen is that eventually, having drawn a feature, t must equal t_1. Therefore we will have that:

- t is the same as it was before the feature was drawn (given).
- a is also the same, because it never changes.
- θ might have changed, but this will only rotate the next steps that will be drawn.

Therefore everything is essentially back to where it started, and so the point will repeat the process it just performed and draw another identical feature, although possibly rotated.

Suppose that it takes n steps to complete a feature. The process will look something like this, with the 6-quiggle for an example:



The red cross marks the starting location of the point, and the green crosses mark other locations where a feature is completed.

So we need to find the smallest natural number n such that $t_{n+1} = t_1$. This means we need to find an integer k such that:

$$t'_{n+1} = t'_1 + 360k$$

We've already established that because successive values of t′ are in arithmetic progression:

$$t'_{n+1} = t'_1 + an$$

$$\therefore\ t'_1 + an = t'_1 + 360k$$

$$\therefore\ n = \frac{360k}{a}$$

Like all computer programs, the program that makes quiggles can only deal with rational numbers. Therefore a must be rational, so the equation above does have a solution.

If a is an integer, then the formula can be simplified further as follows. The a in the denominator must be cancelled by the 360 and the k. If we rewrite a as d × a/d, then we have:

$$n = \frac{360\text{k}}{d \times \frac{a}{d}}$$

Remember that d is the GCD of a and 360, so a/d and 360 have no common factors. Therefore 360 cancels out the d and no more, leaving k as a/d to make n as low as possible, so substituting into the original equation gives:

$$n = \frac{360 \times \frac{a}{d}}{a}$$

$$\therefore\ n = \frac{360}{d}$$

On the $(n+1)^{th}$ step, when t is equal to t_1, the point will travel in the direction θ_{n+1}. This means that successive features will be rotated by θ_{n+1} (the difference between θ_{n+1} and θ_1 is θ_{n+1} because $\theta_1=0$). We can calculate θ'_{n+1} by summing the arithmetic progression of the values of t using the standard formula:

$$\theta'_{n+1} = \frac{n}{2}[2t_1 + (n-1)a]$$

To demonstrate this formula with a z-quiggle, let a and 360 be coprime. Then d = 1 and hence n=360. Substituting this value and t_1 = 1 into the formula gives:

$$\theta'_{n+1} = 180[2 + 359a]$$

Since a and 360 are relatively prime, a is odd, and so the number above is an odd number multiplied by 180, so θ_{n+1} = 180. This means that after completing a feature, the point has overall turned around 180°, and so the next feature is a 180° rotation of the first one, and only two features are formed, which by definition is a z-quiggle. This is a much simpler proof that if a and 360 are coprime, a z-quiggle is formed, but the purpose of the previous chapter was to show what happens during the formation. It is behaviour like the retracing of steps backwards that separates z-quiggles from o-quiggles.

Earlier it was conjectured that if d = 2, then a straight quiggle is formed. If we let d = 2, then n=180, and *a* is even but not a multiple of 4, so let *a* = 2m for some odd integer m. Substituting into our formula again gives:

$$\theta'_{n+1} = \frac{180}{2}(2 + 179 \times 2m)$$

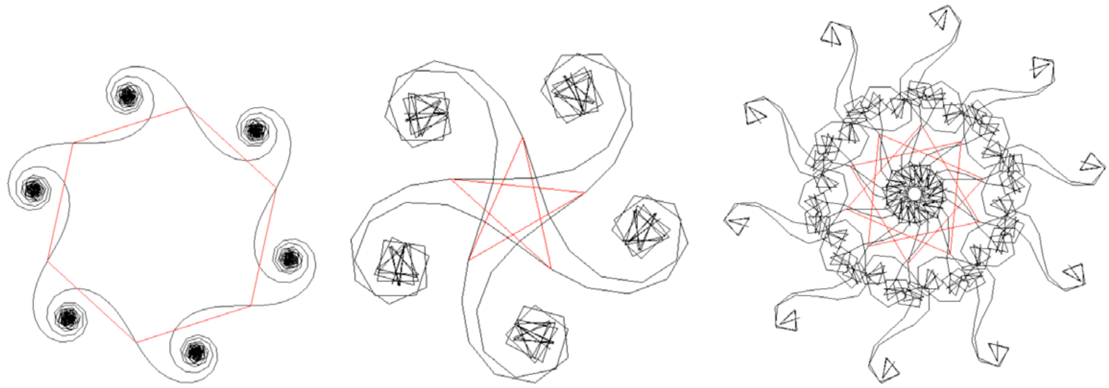
$$= 180(1 + 179m)$$

Since m is odd, θ'_{n+1} here is 180 multiplied by an even number, which is a multiple of 360, so $\theta_{n+1}=0$. This means that the point has not turned while forming a feature, and so the next feature will have the same orientation. Thus one feature after another will be formed in a straight line that continues forever, so our conjecture is proved: if $t_1 = 1$ and *a* is an integer such that the GCD of *a* and 360 is 2, a straight quiggle is formed.

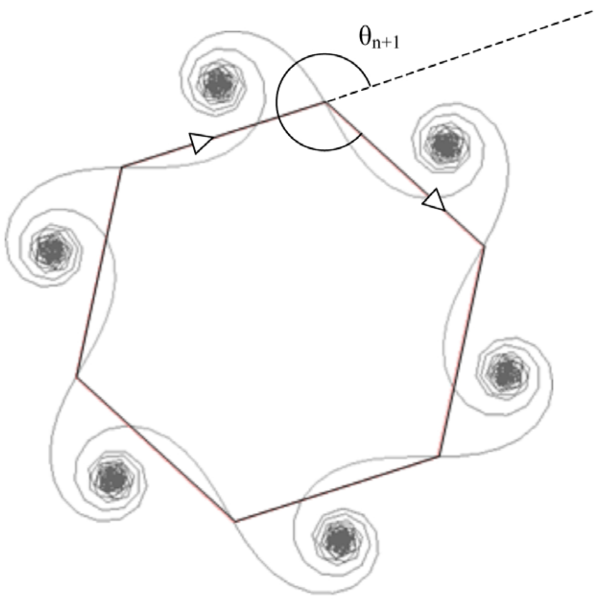
There is one assumption here, though. It might be possible for the point to also return to its original position when forming a feature, and so the feature will be retraced and the full quiggle will only have one feature. This doesn't fall into any category so far, and could be called a mono-quiggle, but I can't find an example of such a quiggle and proving or disproving their existence seems to be an incredibly difficult task beyond this essay. If we assume that the point does not return to its original position, then a straight quiggle is indeed formed.

O-quiggles – the number of features and the exact shape

If d is greater than 2, then things become more complicated, depending on what exactly *a* is. To see what shape the quiggle is, connect the points in the quiggle where features begin and end (done here with red straight lines):

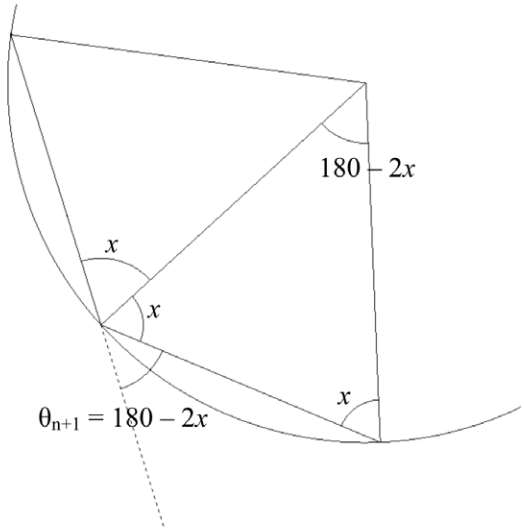


All these lines are the same length, and the angle between any two lines connecting three consecutive features is always the same, so it's a symmetric shape in which all the points lie on a circle·. Overall, the point turns anticlockwise by an angle of θ_{n+1} as it moves from one vertex to another:



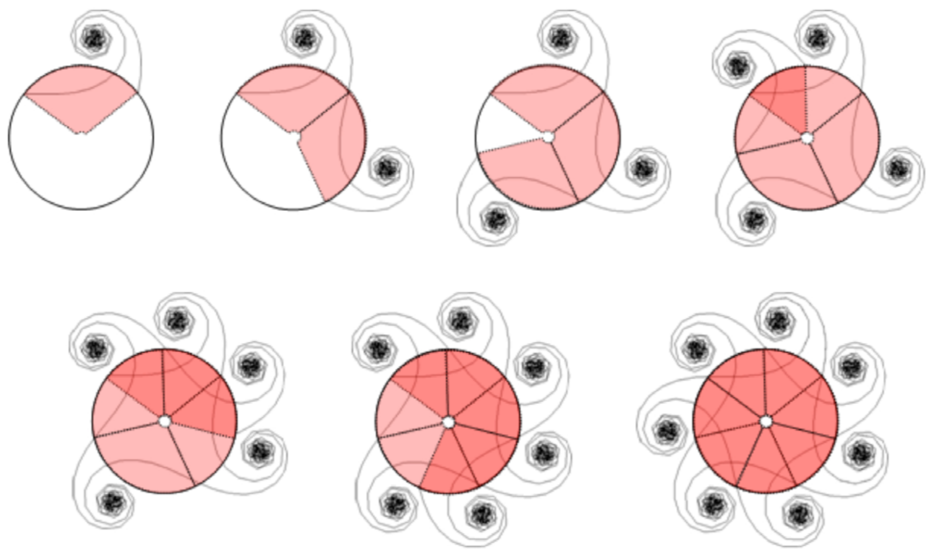
· Again, this is considered a simple, intuitive result, so the proof is in the appendix.

In the diagram below, all the angles labeled x are equal because the shape is symmetrical. Because angles at a line or in a triangle add up to 180, both θ_{n+1} and the angle subtended are equal to 180-2x.



Note that this diagram obviously doesn't work if θ_{n+1} is reflex. In these cases, subtract θ_{n+1} from 360 to find an angle less than 180° that overall the point turns *clockwise*.

Imagine that the circle passing through all the vertices of the shape is a clock, and that a radius connecting the centre and the point's initial position is a hand. Each time a feature is completed, the point turns by θ_{n+1}, and so does the hand of the clock. Let p and r be integers such that after completing p features, the point and the clock hand have both turned through r complete revolutions, so the point is now back where it started (because the clock hand is in its original position and the point is on its end) and also facing the same way. Therefore the quiggle is complete. For example, if p = 7 and r = 2, the process might look like this:



So a quiggle has p features, each subtending an angle of θ_{n+1}, and the point goes round r times, giving the formula:

$$p\,\theta_{n+1} = 360r$$

$$\therefore \quad p = \frac{360}{\theta_{n+1}} r$$

We want to find the lowest integer r such that p is an integer. If we have a z-quiggle, then θ_{n+1} is 180, and the above reduces to p = 2, i.e. z-quiggles have 2 features. If we have a straight quiggle, then p is undefined because we have a division by 0, i.e. straight quiggles never complete a circle. Otherwise, though, we need to put some numbers into our formulae.

For example, let $t_1 = 5$ and $a = 12$:

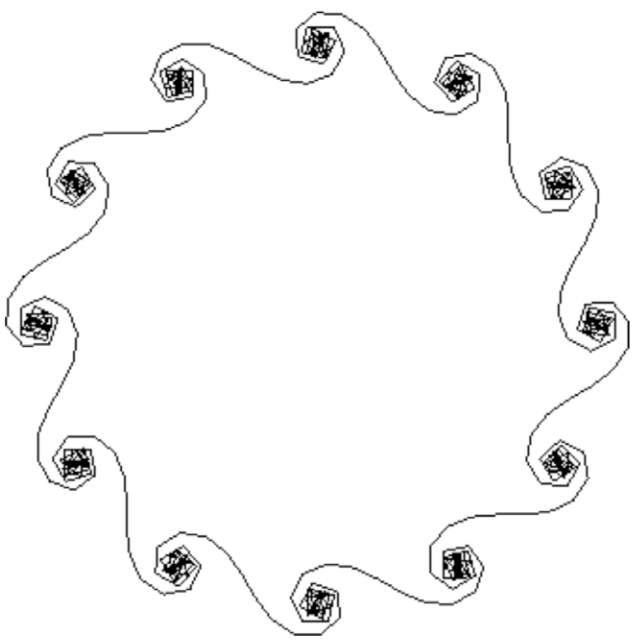
$$d = \mathrm{GCD}(360, 12) = 12$$

$$n = \frac{360}{d} = \frac{360}{12} = 30$$

$$\theta'_{n+1} = \frac{n}{2}[2t_1 + (n-1)a] = 15(10 + 29 \times 12) = 5370$$

$$\theta'_{n+1} = 5370 \equiv 330 = \theta_{n+1} \pmod{360}$$

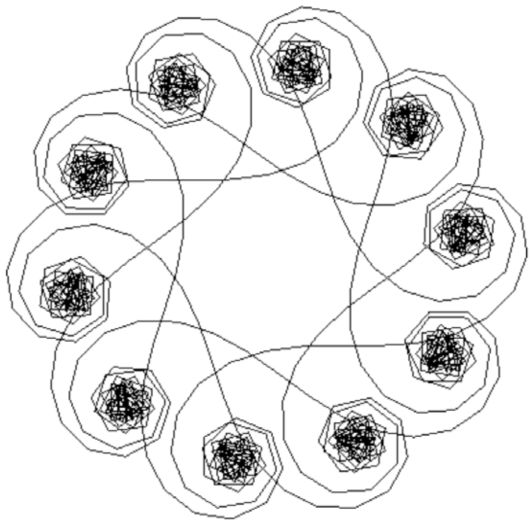
330 is reflex, so subtract θ_{n+1} from 360 to find that the point turns clockwise by 30°. Putting 30 into our formula gives that that p = 12 and r = 1, and indeed the resulting quiggle has 12 features:



It is significant that r = 1, because rearranging our formula when r = 1 gives the equation:

$$\theta_{n+1} = \frac{360}{p}$$

This is the standard formula for the exterior angle of a convex regular polygon with p sides. This means that if r = 1 for an o-quiggle, then the quiggle will correspond to a convex polygon like the one above. If r > 1, then the quiggle will have a regular star-shape. Applying the formulae for the 5-quiggle below gives that p = 10 and r = 3 (after subtracting θ_{n+1} from 360). Therefore we get a quiggle in the shape of a 10-pointed star:



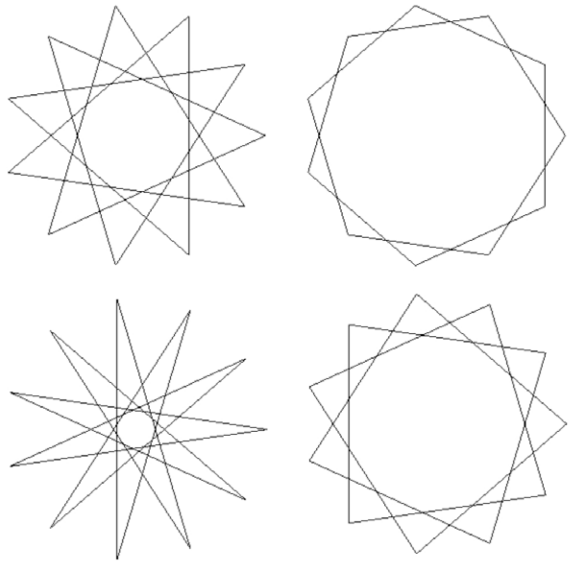
Conclusion

The investigation is complete. We now have a full process for predicting, given a and t_1, what shape a quiggle will take. Note that throughout this essay, I have stuck to positive integers to simplify the process. But the formulae in this essay do not require this. Using negative numbers would be strange, and probably pointless, but it doesn't hurt – it simply means clockwise instead of anti-clockwise. Using fractions can also make calculations more difficult, but it can be done and will often produce great results. Only irrational numbers are impossible, because a computer cannot calculate with them. Therefore only the first formula has to change (by omitting d) to generalise the process of prediction for all rational numbers. The process is as follows:

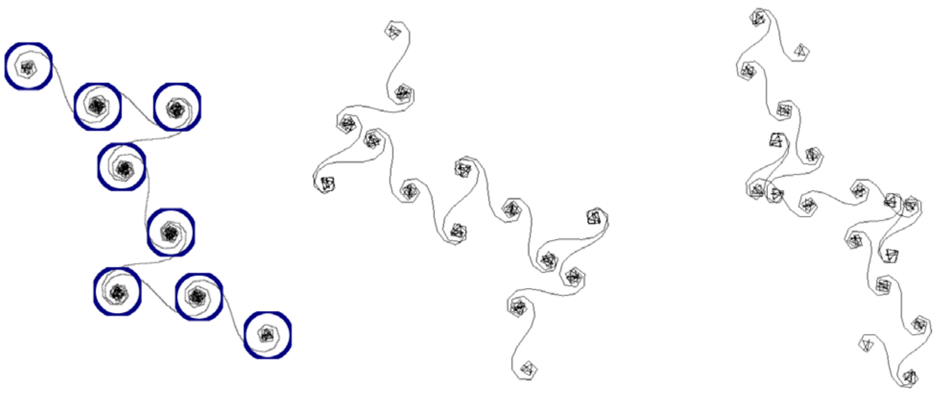
1. Solve the equation $n = \frac{360k}{a}$ for the lowest positive integers n and k to find out how many steps it takes to make a feature.
2. Evaluate $\theta'_{n+1} = \frac{n}{2}[2t_1 + (n-1)a]$ and take the remainder modulo 360 to find the anticlockwise angle of rotation between successive features. If the angle is greater than 180, subtract it from 360 to find the clockwise angle of rotation. If the angle is 0, then the quiggle will be a straight quiggle (or possibly a one-featured mono-quiggle?).
3. If the angle is not 0, take whichever angle is less than 180, be it clockwise or anti-clockwise, call it x, and use it to solve the equation $p = \frac{360}{x} r$ for the lowest positive integers p and r. p will be the number of features: if it is 2, we have a z-quiggle, and if it is more, we have an o-quiggle. If r = 1 for an o-quiggle, the quiggle will be in the shape of a regular convex polygon. Otherwise, it will have a regular star-shape with intersecting lines.

However, this is not the end of investigating quiggles altogether. Many questions remain. For example, the existence of mono-quiggles is still an open issue. I never had to deal with the actual position of the point or the distances it moved to answer my research question: working with angles was sufficient. Intuition makes me doubt their existence greatly, but the point moves in such a complex way that more advanced mathematics is probably required to prove or disprove this conjecture.

There are also questions about deeper details of appearance. For example, predicting whether an o-quiggle will be convex or star-shaped is easy. Predicting exactly what a star will look like is more interesting. The diagram on the next page shows four truly different 11-pointed stars. Based on the exact value of r, how can one predict which star shape a quiggle will be?



The questions continue. What about the number of 'knots' in a quiggle, where the lines get bunched up when t is in the region of 180? z-quiggles with an integer value of a seem to always have a+1 knots. Why?



a = 7, knots = 8 a = 13 a = 17

I hope I have sparked your interest in quiggles.

Bibliography

[1] University of GothenBurg "Roller Coaster Loop Shapes" Department of Physics, University of GothenBurg, Sweden. http://physics.gu.se/LISEBERG/eng/loop_pe.html (accessed May 3, 2010)
Image replaced for copyright reasons. Image by Andreasr24 (http://commons.wikimedia.org/wiki/File:Loopen_p%C3%A5_TusenFryd.jpg)

[2] Walton, Desmond. "The Euler/Cornu spiral, Clothoid" SciTopics. http://www.scitopics.com/The_Euler_Cornu_spiral_Clothoid.html (accessed May 3, 2010)

Appendices

Proofs of standard or simple theorems in essay

Theorem 1:

$$t_{i-j} + t_{i+j} = 360, \text{ where } t_i = 180 \text{ and i, j are positive integers, } j < i$$

(Used in the formation of z-quiggles, when explaining why the point retraces, pg. 13)

We know already that:

$$t'_{n+1} = t'_1 + an$$

Or, more conveniently to prove this theorem:

$$t'_n = t'_1 + a(n-1)$$

Substituting n=i-j and n=i+j gives the two equations:

$$t'_{i-j} = t'_1 + a(i-j-1)$$

$$t'_{i+j} = t'_1 + a(i+j-1)$$

Adding these two equations gives:

$$t'_{i-j} + t'_{i+j} = 2t'_1 + 2a(i-1)$$

$$= 2t'_i$$

Taking the remainder modulo 360 and noting that $2t_i = 360$ gives the result required.

Theorem 2:

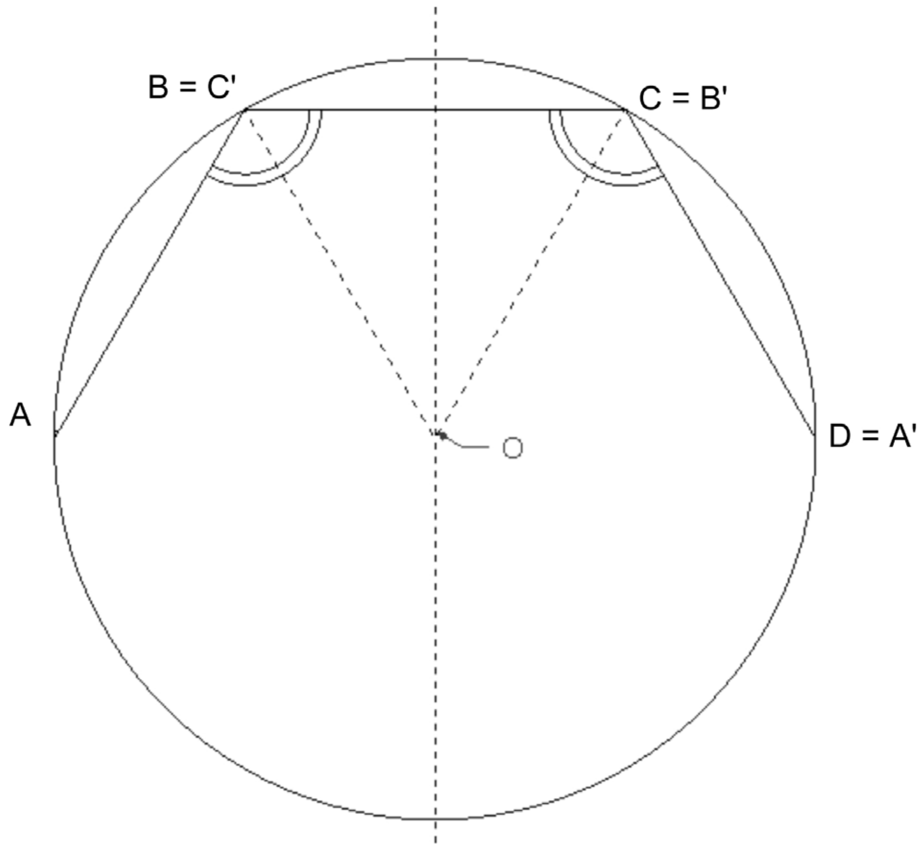
A set of equal line segments connected in a cycle such that all the angles between two adjacent line segments are equal will have all its endpoints lying on a circle

(Used in the chapter on O-quiggles when talking about connecting the endpoints of features, pg. 18)

Although this is not the simplest theorem to prove, it seems obvious after a bit of thought – the shape described is a perfectly symmetric cycle and to imagine the points not lying on a circle seems absurd, which is why it is left to the appendix.

Take any 4 points ABCD in the cycle. All non-degenerate triangles have a unique circumcircle, so call the circumcentre of ABC O. BO and CO are radii of the circumcircle, so BCO is an isosceles triangle. ABCD is a symmetric shape, so it must have a line of symmetry, and since BCO also has a line of symmetry, and it lies in the middle of the shape, the two lines of symmetry must be the same. Also note that O must lie on this line of symmetry, as O is part of the isosceles BCO.

The image A′B′C′ after reflecting ABC in the line of symmetry will coincide with BCD. BC and O have not moved in this reflection (although B and C have switched places) so three points defining the original circle are unmoved, and hence the circle cannot have moved either, but it still lies on the image of ABC, so it lies on BCD. Therefore ABC and BCD have the same circumcircle, and hence ABCD has a circumcircle. This is all shown in the diagram below:



Now we have an initial case for a proof by induction of our theorem, and next we need an inductive step. Assume that it has been proven that any k or less adjacent points must lie on a circle for some positive integer k (our initial case is k=4. The cases k=1, 2 or 3 are trivial). To prove that k+1 points must also lie on a circle, consider the set of adjacent points $P_1P_2P_3 \ldots P_kP_{k+1}$. It has already been proven that the points $P_1P_2P_3 \ldots P_k$ all lie on a circle, as well as $P_2P_3 \ldots P_kP_{k+1}$, and the smaller intersection of these two sets $P_2P_3 \ldots P_k$. The circumcircles of $P_1P_2P_3 \ldots P_k$ and $P_2P_3 \ldots P_kP_{k+1}$ must both coincide with the circumcircle of $P_2P_3 \ldots P_k$, because the latter set is a subset of the former sets. Thus the circumcircles also coincide with each other and all k+1 points must have a single circumcircle.

About the program: I created the program for generating quiggles using a programming tool called Game Maker, designed by Mark Overmars. The code to make the program is too lengthy to include here, so email me at mojaki2@yahoo.com (I hope I will still have access to the address when you read this) if you want to receive a copy of the program. It is a self-executable file (meaning that you will need nothing else to run it) but it is only compatible with Windows. It is small enough for email.

