# Mathematics higher level

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| Mark range: | 0 - 14 | 15 - 29 | 30 - 40 | 41 - 53 | 54 - 66 | 67 - 78 | 79 - 100 |

## Calculus overall

| Grade: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
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| Mark range: | 0 - 13 | 14 - 27 | 28 - 39 | 40 - 52 | 53 - 66 | 67 - 78 | 79 - 100 |

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## Higher level paper three discrete mathematics

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|---|---|---|---|---|---|---|---|
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## Higher level paper three calculus

| Grade: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
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## Higher level paper three sets, relations and groups

| Grade: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
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| Grade: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Mark range: | 0 - 9 | 10 - 19 | 20 - 28 | 29 - 33 | 34 - 37 | 38 - 42 | 43 - 50 |

# Higher level internal assessment

## The range and suitability of the work submitted

Clearly some schools had an exploration process in place that guided the students and encouraged them to write explorations in topics of their own interest. These schools should be commended.

On the other hand, a significant number of candidates submitted what they themselves called a "research report" which simply consisted of transcribing mathematics and copying images from online sources, textbooks and / or video clips. In a number of cases, candidates wrote about topics way beyond the syllabus, or modelled a situation from another discipline e.g. Music, Physics, Economics without doing the topic justice. It is difficult in such cases for students to write an exploration that meets the aims of the IA within the page limit. As stated in the guide "The final report should be approximately 6 to 12 pages long. Students should be able to explain all stages of their work in such a way that demonstrates clear understanding. While there is no requirement that students present their work in class, it should be written in such a way that their peers would be able to follow it fairly easily."

Some candidates produced explorations that were too ambitious, in topics that would have been more appropriate for an extended essay. Unfortunately, this was the case with an increased number of candidates who wrote explorations about Fourier and Laplace transforms.

An exploration that simply relays published findings is not likely to achieve high levels. The re-occurrence of staid, overly-subscribed topics which provide little potential for personal connection, such as the "SIR" model, projectile motion and "RSA encryption" was also noted.

There was evidence to suggest that some students are not adequately guided in their choice of topic. Although the exploration topic is chosen by the student the teacher needs to provide guidance to ensure that the topic explored is focused and lends itself to the aims of the Internal Assessment.

## Candidate performance against each criterion

### Criterion A

It is a pity that the highest achievement level in this criterion seemed to be inaccessible for a larger number of students. It seemed that some schools encouraged students to include a contents page and a research question. In some cases, students ended up writing a piece of work that read more like a chapter out of a text book which would have been difficult for an average HL student to understand.

### Criterion B

The candidates' mathematical presentations were generally good. Common problems were not defining terminology which is either beyond the HL syllabus or which comes from other subject areas.

### Criterion C

This criterion continues to be too generously marked by teachers. A statement or rationale alluding to interest in the topic does not constitute personal engagement. Evidence for personal engagement needs to be shown within the work, including the mathematics attempted. When topics are presented in a text-book style they do not reflect personal engagement.

### Criterion D

Critical reflection throughout the exploration continues to be a rare find. Often a summative reflection at the end of the exploration was provided and although this does not necessarily preclude high achievement levels it makes it more difficult to reach these levels. It should be noted that when the mathematics is at a limited level and good understanding is not demonstrated then it would be difficult to achieve critical, or even meaningful, reflection. Critical reflection in explorations should be ongoing, and act as a stepping stone from one part of the exploration to another.

### Criterion E

In general, there was a large variety of mathematical content. The greatest challenge for students appears to be to choose a topic with mathematics at an appropriate level. Students who chose to write research reports on topics beyond the HL syllabus struggled when trying to explain the mathematics from one step to another, making it difficult to gauge the level of understanding. Unfortunately, many times the teachers recognized this as sophisticated and rigorous and awarded top levels. The key words in this descriptor are "demonstrating understanding". Unfortunately, a number of times, errors were found in students' work that were not identified by the teacher.

## Recommendations and guidance for the teaching of future candidates

Unfortunately, many shortcomings in the student work is due to the lack of guidance which is often a reflection of the teacher's lack of understanding. This is clear evidence that teachers do not read the Teacher Support Material or the subject reports to understand the IA objectives.

The selection of an appropriate theme that allows the opportunity for the use of Mathematics that is commensurate with the level of the course should be emphasised. Students should be advised to try and present a personal example and try to obtain a solution rather than reproduce general information found in sources, this also helps in criterion C. The student should always explain why every step is done and how results are obtained. More attention should be paid to critical thinking and applications of every math topic while teaching with references to possible applicability of use in the IA. A list of 'good' and 'not so good' exploration topics could be shared with students and at the start of the formal process students can brainstorm using these ideas. The practice of having students read math articles in academic journals needs to be encouraged.

Teachers must emphasize that the explanation is for peers. In most cases candidates seem to focus on explaining every step on the calculations, but do not link these calculations to the development of the exploration.

Although the exploration should be introduced as early in the course as possible, the actual process should be delayed until a fair amount of the syllabus content has been covered. Teachers should invest time in going over the criteria descriptors with students to ensure that students thoroughly understand the expectations. One way of doing this would be to use explorations from the Teacher Support Material with students. There was evidence to suggest that students were not always given adequate feedback on a first draft. Students should also be advised to proof-read their work before submission.

Teachers should annotate student work with appropriate comments to indicate how the individual criterion descriptors where interpreted. Merely writing, A+ or E-, is not enough. Evidence of marking the mathematics should also be in evidence. This includes tick marks to indicate correct work, identification of errors, annotations and comments to explain where and how the achievement levels were awarded. The moderator's role is to confirm the teacher's marks but where annotations and comments are missing the moderator will have to mark the work without having any background information and very often it is less likely that a moderator can confirm all the achievement levels awarded. Comments in the form of sticky notes automatically become detached when the work is scanned and attached separately at the beginning of the work. Their use is not helpful in moderation and it should be discouraged.

# Higher level paper one

## General comments

The paper was generally well done by candidates with many showing good understanding of all aspects of the course. There were plenty of good answers to each of the questions indicating they were set at an appropriate level to meet the assessment objectives.

## The areas of the programme and examination which appeared difficult for the candidates

The areas that presented most difficulty were: counting principles, complex numbers (including de Moivre's theorem), sums and products of roots, lines and planes and solving quadratic inequalities.

## The areas of the programme and examination in which candidates appeared well prepared

The calculus sections were generally very well done, including implicit differentiation and integration by parts. It was pleasing to see plenty of well-structured proofs by induction, though this was not universally well done. The candidates also produced many good answers when asked to solve a system of linear equations.

## The strengths and weaknesses of the candidates in the treatment of individual questions

### Question 1

There were a few problems for some candidates in placing $\mathrm{P}(A' \cap B)$ on the Venn diagram in part (a). Most candidates though were able to use the formula correctly in part (b), though some still took independent to mean mutually exclusive.

### Question 2

This was largely well done. A few candidates used permutations, and a few did not know how to calculate $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$ without a GDC. The majority who managed a correct solution to part (b) did it by subtracting from the total though a few were successful in adding up three sets of combinations.

### Question 3

(a) There were relatively few candidates who obtained full marks in this part. The most common error was not drawing a curve which was concave down throughout the range.

(b) The majority of candidates understood what was being asked in this question.

(c) Both parts were well done. Future candidates should be reminded that $y =$ is not an expression for $g^{-1}(x)$.

## Question 4

Part (a) was poorly understood by many candidates, who did not appreciate what was being asked. Part (b) though was well done even by weaker candidates who largely completed the algebraic manipulation without error. There were approximately equal numbers using the methods of row reduction and elimination.

## Question 5

Part (a) was achieved successfully by most candidates. In part (b) many struggled to relate the angle being obtuse to the cosine of the angle being negative. Finding the solution to a quadratic inequality was also poorly done by many candidates.

## Question 6

This question was well done by many candidates. A few marks were lost by students not setting out the case for n=1 clearly. A statement that 1=1, is not sufficient and the candidates were expected to show both the left-hand side and the right-hand side were equal to 1, and have a concluding statement that the statement was therefore true for n=1.

The statement 'Let n=k' was also seen, as was the statement 'suppose n=k is true'. It is important to use the language correctly to ensure the proof is rigorous.

## Question 7

Part (a) was well done by most candidates. Those who did not use implicit differentiation had more problems in the second part. In part (b) many candidates did not realize the importance of the point P being on both curves and simply showed that the product of the two derivatives (in terms of $x$ and $y$ ) was equal to -1, which was not awarded any marks.

## Question 8

Most candidates managed to achieve some marks on this question. Difficulties arose though when they forgot to use the formulae for the sum and product of roots and tried instead to multiply out the factors. There were though plenty of excellent answers to this part.

## Question 9

There were plenty of good solutions to part (a) with candidates largely untroubled by the presence of $b$.

A few gave the vector rather than the Cartesian form for the plane.

Part (b) proved more difficult with some candidates failing to realise that they needed to use the perpendicular vector obtained in part (a) as the vector parallel to the required straight line.

Some candidates were still writing $L =$ rather than $\boldsymbol{r} =$.

Part (c) proved to be one of the most difficult questions on the paper. There were though plenty of excellent solutions which followed a variety of different methods.

## Question 10

The method for finding the integral of a product of a trigonometric function and an exponential function was known by the majority of candidates and was successfully completed by many.

Part (b) was largely successful. Candidates used the hint of 'hence' and converted $\cos^2 x$ using the double angle formula. Those who tried to use parts again and then convert using the double angle formula later in the process were only occasionally successful.

In part (c) there were a few errors in the derivatives. Some candidates converted to a double angle form before differentiating. Many of these still reached the correct answer by converting back again once the differentiation was completed. About half the candidates who found the first solution were successful in finding the second.

Part (d) was successfully done by many candidates, the main errors being due to not realizing the values of $\cos 3\pi$ and $\cos \pi$ .

### Question 11

There were some good solutions to part (a), but overall this was one of the weakest questions, and de Moivre's theorem is something that may need more focus in the future.

(b)(i) required use of the identity that $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ this was well used by many of the candidates attempting this question.

(b)(ii) was successfully completed by many candidates.

(b)(iii) This was only completed by a few candidates, one of the difficulties was that candidates needed to work out an expression for $\cos\left(\frac{5\pi}{12}\right)$ which was missed by many.

## Recommendations and guidance for the teaching of future candidates

There should be more emphasis on the structure of proof, considering LHS & RHS separately when dealing with 'show that' questions or proof by induction.

When dealing with functions, students should carefully consider the given domain and give answers accordingly. This is particularly the case when dealing with trigonometric and, on this paper, complex numbers.

It is important students use the correct notation, so when writing a vector equation of a line, it is essential to write it as $\boldsymbol{r}$ = ….and when finding the inverse of a function, the solution should be written with the appropriate notation (not as $y$ = ).

Students need to be reminded of the implication of the word 'hence'.

The following topics were generally poorly done and so probably need more emphasis when teaching the course:

Solving quadratic inequalities, counting techniques and the evaluation of $\binom{n}{r}$ without a calculator, complex numbers and De Moivre's theorem.

# Higher level paper two

## General comments

There was a good selection of excellent responses to this paper, although there is still a problem with some candidates not using their graphical display calculator effectively or efficiently. This can result in lengthy algebraic manipulations for some questions for very little gain and can result in too much time being spent on relatively short questions.

## The areas of the programme and examination which appeared difficult for the candidates

- Finding an expression for a function given the second derivative and two boundary conditions.
- Showing every step in a differentiation from first principles, using appropriate notation.
- Using a graphical display calculator efficiently to estimate the mean and standard deviation from a grouped frequency table.
- Ensuring that all the required information is shown on a graph sketch, including equation of asymptotes.
- Understanding the graphical significance of a repeated linear factor in a polynomial function.
- Using a graphical display calculator efficiently to find a point of inflexion.
- Using the compound angle formula for tangent.
- Solving problems in context involving the areas of segments.
- Using the symmetry of the graphs of inverse functions to sketch graphs and solve problems.
- Interpreting a graph in relation to a context given in the question.

## The areas of the programme and examination in which candidates appeared well prepared

- Using the formulae for the general term and the sum to infinity of a geometric sequence.
- Differentiating a function using the quotient or the product rule.
- Applying the factor theorem.
- Find the constant term in an expression involving a binomial expansion.
- Finding the mean and standard deviation of a normal distribution, given information about associated probabilities.
- Solving simple problems using the Poisson distribution.

## The strengths and weaknesses of the candidates in the treatment of individual questions

### Question 1

Generally, very well done. A minority of candidates made a sign error in finding the value of the common ratio. Many did not give their answer to part b) in the correct form, as an exact value or rounded to 3 significant figures.

## Question 2

Most candidates understood clearly that the question required them to integrate twice. However, a surprising number were not able to integrate correctly, making errors with the rational power and the negative power. Few candidates were completely successful in recognising the need for two different constants of integration.

## Question 3

Generally, well done. This was a standard question and many candidates were clearly very confident. However, it was common for a mark to be lost due to a lack of accuracy, resulting from the use of inverse normal values rounded prematurely to 3 significant figures. Some candidates did not recognise the continuous nature of a normal distribution when answering part (b). There was a small minority who did not use the inverse normal distribution at all.

## Question 4

Generally, very well done. The most common error was to evaluate $3^3$ as 9. A variety of methods was seen, and it was surprisingly common to see the entire expansion before the required term was identified.

## Question 5

Most candidates appeared to familiar with differentiation from first principles and were able to present a solution which was almost complete. However, it was common to see minor errors with powers in the intermediate steps, and to see the crucial step of cancelling *h* not stated explicitly.

## Question 6

Part a) was often well done, but many candidates did not appreciate the graphical significance of a repeated linear factor. Part (c) was often answered successfully using long division methods, especially by those who could not answer part (b).

## Question 7

This question was surprisingly problematic, and it was disappointing to see comparatively few good solutions. Many candidates tried to find the angles in the sectors using right angles trigonometry by splitting the distance AB in an arbitrary way or by assuming a right-angled triangle with sides 5 and 7. Some found the correct angles in degrees and did not convert their answers into radians before using the formula for the area of a sector. It was not uncommon to see candidates using the equations of circles, which are not on the syllabus, and then using integration techniques. Some of these attempts were excellent and earned full marks, but this was rare.

## Question 8

This proved to be a challenging question for many, although most candidates recognised the invariant point A and many clearly knew the shape of a rational function. It was disappointing that correct sketches often lost marks because the equations of the asymptotes were not included. In part b), many candidates obtained correct answers though faulty reasoning, such as deducing that $b=3$ and $c=-4$ from information that $\frac{-c}{b}=\frac{4}{3}$.

## Question 9

Part (a) was usually well answered.

In part (b), many candidates attempted to differentiate again in order to find $f''(x)$, spending a lot of unnecessary time on a complicated algebraic process. This approach was rarely successful. The best solutions involved the effective use of the graphical display calculator to find a turning point on the graph of $y = f'(x)$, or to solve the equation $f''(x) = 0$ using a graph or using an equation solver.

It was rare to award full marks for the sketches required in part (c), for a multitude of reasons. Some candidates sketched the function $f$ outside the given domain, and many did not clearly identify the asymptotes by considering the nature of the function. The calculator cannot be relied on to show asymptotic behaviour clearly, and the display must always be combined with the candidates' knowledge of functions. The axis intercept was sometimes not given to an appropriate degree of accuracy. Some candidates appeared to draw the reciprocal graph rather than the inverse. Of those who understood that the inverse graph would be a reflection in the line $y = x$, those who drew both graphs on one set of axes were more successful.

In part (d), many candidates attempted to find an expression for the inverse function instead of recognising that they just needed to find the intersection of $y = x$ and $y = f(x)$.

## Question 10

In part (a)(i), a large number of candidates failed to understand the discrete nature of the Poisson distribution, and found $\mathrm{P}(X \le 60)$. Part (a)(ii), by contrast, was well answered by most.

In part (b), a surprising number of candidates could not find the correct midpoints, and it was common to see the use of 45, 55 etc… Many candidates went through a complicated process of finding an estimate for the mean and standard deviation, often introducing errors. Some candidates worked with a total frequency of 180 rather than 120. The most efficient and effective way to solve this problem was to enter the frequency table into the calculator.

In part (c), there was a variety of acceptable reasons. However, many candidates were not awarded this mark because they did not back up their reason with numerical evidence. The most common example was to state that the mean and the variance of Willow's data were not close together, without ever stating the value of the variance.

Part (d) was not well solved, with many candidates misinterpreting "more than 10" as "at least 10". Also, it was common for candidates to attempt to solve $\mathrm{P}(X = 10) = 0.01$ rather than $\mathrm{P}(X \le 10) = 0.01$.

In part (e), there were some excellent solutions which showed clear cancelling of $\lambda$ to arrive a solution. Many candidates used a numerical value of $\lambda$, demonstrating a misunderstanding of the nature of the question.

## Question 11

Most candidates answered part (a) effectively and succinctly using tan. Some candidates did not position the angles correctly against the base of the rectangle, and some approached the question using sin or cos, resulting in more complicated but correct expressions.

Part (b) was either done very well or very poorly, depending on whether the candidates understood that they had to use the compound angle formula for tan rather than assuming that $\tan(A+B)=\tan A+\tan B$.

Most candidates were able to sketch the graph correctly in part (c), but very few were able to relate it back to the original problem and recognise that $q<9$ using the information given in the question stem.

## Recommendations and guidance for the teaching of future candidates

Ensure that all parts of the syllabus have been taught; this year a standard problem involving areas of segments appeared to be unfamiliar to many candidates and the relationship between the graphs of a function and its inverse was not widely understood. In addition, the significance of a repeated linear factor in a polynomial function should be emphasised.

Ensure that candidates know how to use their graphical display calculator efficiently. Examples include finding the coordinates of a point of inflexion and estimating the mean and standard deviation from a grouped frequency table.

Insist that students practise drawing clear, labelled sketches of graphs. The command term "Sketch" requires "a general idea of the required shape of relationship and should include relevant features."

# Higher level paper three discrete mathematics

## General comments

This examination was an accessible test of a candidate's knowledge of discrete mathematics. Although a good percentage of candidates demonstrated sound knowledge of the syllabus, it was also apparent that some candidates were not sufficiently prepared and performed poorly.

## The areas of the programme and examination which appeared difficult for the candidates

- Showing that a positive integer $N$ in base 5 is exactly divisible by $4$ if and only if the sum of the digits is exactly divisible by 4 and showing that $(x12x)_5$ cannot be exactly divisible by 4.
- Formally showing that $I_n$ satisfies the second-degree recurrence relation $I_{n+2} - 2.2I_{n+1} + 1.2I_n = 0$.
- Stating two limitations of a proposed model described by a given recurrence relation.
- Using Dijkstra's algorithm to find the minimum travel time between two vertices in a weighted graph $G$.
- Finding the minimum time taken to travel around a weighted graph $G$ using each edge at least once (Chinese postman problem).
- Performing base $5$ multiplication.

## The areas of the programme and examination in which candidates appeared well prepared

- Using the Euclidean algorithm to find the greatest common divisor of two numbers.
- Recognising that $ax + by = c$ has solutions provided that $\gcd(a,b) \mid c$.
- Stating appropriate initial conditions for a proposed model described by a recurrence relation.
- Stating, with a reason, whether a graph $G$ has an Eulerian circuit.
- Using Kruskal's algorithm to find a minimum spanning tree for a weighted graph $G$.
- Performing base $5$ addition.
- Attempting to solve a linear Diophantine equation.
- Attempting to solve a second-degree recurrence relation.

## The strengths and weaknesses of the candidates in the treatment of individual questions

### Question 1

Part (a), a routine question, was very well done. Only a small number of candidates were not able to correctly apply the Euclidean algorithm to find that $\gcd(77,105) = 7$.

In part (b), a good number of candidates recognized that the linear Diophantine equation has a solution when $c$ is a multiple of $7$ (or equivalent statement). A few candidates stated that the equation has a solution only when $c = 7$.

In part (c), the majority of candidates attempted to solve $11x+15y=1$ rather than using their work from part (a) to solve $77x+105y=7$ and then divide by $7$. A large number of candidates found that $x=-4, y=3$ is a solution to the linear Diophantine equation. However, a few of these candidates gave this as their final answer rather than going on to determine the general solution. Some candidates gave general solutions containing sign errors.

## Question 2

Part (a) (i), involving base $5$ addition, was generally well done. A few candidates performed base $10$ addition only while others made arithmetic errors converting to base $10$ before adding and then converting back to base $5$. Part (a) (ii), involving base $5$ multiplication, proved a bit more difficult for a number of candidates. Some candidates performed base $10$ multiplication only while others made arithmetic errors converting to base $10$ before multiplying and then converting back to base $5$. A few candidates gave $(5142)_5$ as their final answer.

Part (b), a divisibility proof, was the most difficult question part on the paper. Only the very best candidates displayed a good grasp of what is required to prove the result. These candidates typically were able to exploit the result that $5^2 \equiv \ldots \equiv 5^n \equiv 1(\bmod 4)$.

Part (c) was more accessible to candidates with some weaker candidates offering clear and correct arguments as to why $(x12x)_5$ cannot be exactly divisible by 4.

## Question 3

Part (a) was not well done. From the worded information, candidates were expected to formulate $I_{n+2}-I_{n+1}=1.2(I_{n+1}-I_n)$ leading to the required result $I_{n+2}-2.2I_{n+1}+1.2I_n=0$.

In part (b), nearly all candidates stated appropriate initial conditions for the proposed model. These could be stated in part (a) or part (c) to be awarded the mark.

In part (c), most candidates attempted to solve an auxiliary equation with a number obtaining $I_n=50(1.2)^n-50$. A few candidates found incorrect solutions to a correct auxiliary equation. The most common error of this type was determining that $\lambda=1$ only and treating the situation as the equal (repeated) roots case.

In part (d), a reasonable number of candidates used their GDCs successfully to solve either the required equation or their equation for $n$. A few candidates used by-hand algebra to determine that $(1.2)^n=101$ before finding the correct answer. A few candidates gave $n=25$ as their final answer.

In part (e), most candidates offered at least one reasonable limitation of the model. The two most popular limitations given were that $I_n$ are not all integers and the assumption that people do not recover from the virus or get reinfected with the virus.

## Question 4

Part (a) was well done with most candidates stating that $G$ does not have an Eulerian circuit because it has vertices of odd degree.

Part (b) was quite well done with most candidates using Kruskal's algorithm correctly to find the minimum spanning tree, stating its total weight and indicating the order in which the edges were added. A few candidates identified 7 correct edges and a correct total weight but not the required edge order. A handful of candidates identified 6 correct edges and did not indicate the required edge order.

In part (c), a substantial number of candidates started using Dijkstra's algorithm but did not carry out the algorithm fully to show conclusively that the minimum travelling time from $\mathrm{A}$ to $\mathrm{E}$ is 13 minutes. A few candidates attempted other algorithms while others merely stated the result by inspection.

In Part (d), a number of candidates identified that there are 4 vertices of odd degree, listed at least two possible pairings of odd vertices, found that $\mathrm{A} \rightarrow \mathrm{F}$ and $\mathrm{B} \rightarrow \mathrm{D}$ have minimum weight and determined a minimum patrol time of $139$ minutes. A few candidates determined that $\mathrm{A} \rightarrow \mathrm{D}$ has weight $21$ rather than $20$. Some stated that road $\mathrm{BD}$ needs repeating rather than roads $\mathrm{BC}$ and $\mathrm{CD}$ while others neglected to mention that $\mathrm{AF}, \mathrm{BC}$ and $\mathrm{CD}$ are repeated roads. Some candidates used other graph algorithms, while others made simple arithmetic errors when calculating the total weight of the edges.

## Recommendations and guidance for the teaching of future candidates

- Emphasize the difference between a specific solution and a general solution to a linear Diophantine equation.
- Examine a variety of divisibility proofs in base $10$ extending this to other bases.
- Practice formulating recurrence relations from worded information.
- Discuss the limitations of a proposed model described by a recurrence relation.
- Showcase use of a GDC to solve equations of the form $u_n = f(n)$ for integer values of $n$.
- Show different graphical and tabular ways that display clear use of Dijkstra's algorithm.
- Provide opportunities for candidates to select and use appropriate graph algorithms in worded contexts.
- Emphasize the need to show appropriate reasoning and clear methods/steps leading to the correct answer in a "show that" question.
- Advise candidates to read examination questions carefully and write legibly as this will greatly help examiners with their marking.
- Continue to dissuade candidates from using graph paper in examinations. Scanned scripts containing graphs/diagrams on graph paper are often difficult to mark.

# Higher level paper three calculus

## General comments

Although the feedback from most teachers indicated that this paper was of similar standard as previous examinations, many candidates found the paper challenging. It was also evident that a significant number of candidates were not sufficiently prepared to answer questions about all parts of the Calculus option and performed poorly.

## The areas of the programme and examination which appeared difficult for the candidates

- Conditions for convergence for limit comparison test for numerical series.
- Use of ratio test to study the convergence of a series of functions.
- Understanding of the Fundamental Theorem of Calculus.
- Application of differentiation techniques involving implicit differentiation and higher derivatives.
- Understanding of isoclines.

## The areas of the programme and examination in which candidates appeared well prepared

- Routine application of L'Hôpital rule.
- Routine application of Euler's method.
- Solving linear and homogeneous first order differential equations.

## The strengths and weaknesses of the candidates in the treatment of individual questions

### Question 1

There was a mixed response to this first question. In part (a) candidates who recognized the need to use harmonic series successfully applied the limit comparison test to obtain the required result. Unfortunately, many failed to state the condition satisfied by the limit to conclude divergence which prevented them from achieving full marks in this question. However, a significant number of candidates showed difficulties in recognizing that the series given has the same behaviour as the harmonic series. The candidates that attempted to prove that the series was convergent had difficulties in dealing with the limit obtained and showed lack of understanding of the limit comparison test (LCT) in a case where the theorem is inconclusive, as they had attempted to compare to a convergent series, but concluded the series to be divergent.

In part (b), many candidates made a good attempt at the ratio test, with a number of candidates having difficulty in obtaining the value of the limit. Many lost the reasoning mark for not stating that the limit obtained was always less than 1 for all real values of $x$.

### Question 2

Most candidates achieved some marks on this question by recognizing the need to apply L’Hôpital Rule and differentiating both terms once correctly. However, candidates often had difficulty in computing the second derivative accurately as many struggled with the application of product rule. Response to part (b) showed a lack of understanding of the required concepts and skills by candidates. Only a minority of candidates recognized the application of Fundamental Theorem of Calculus. A number of candidates gained follow through marks in (b).

### Question 3

In part (a), candidates who used implicit differentiation generally scored well in this question. A large proportion of candidates who chose to make $y$ the subject did poorly in this part of the question and in general lost valuable time in the paper. Once again a number of misconceptions were identified, namely the meaning of $\frac{d^2y}{dx^2}$, mistakenly taken as $\left(\frac{dy}{dx}\right)^2$ or $y$ taken as independent of $x$.

In (b) candidates who saw the connection to part (a) usually did well. However, some candidates seemed unfamiliar with Taylor polynomials, trying to integrate to find an expression for $y$ first. 3(b) and (c) was generally well attempted with a number of candidates gaining follow through marks from (a) and (b) respectively. Only candidates with answers to part (b) were able to score in part (c).

### Question 4

In part (a) many candidates scored well and showed good technology skills. However, some candidates seemed unsure how to begin and were unfamiliar with Euler's method.

Part (b) was well answered but many candidates lost a mark for omitting the modulus sign in their final answer. 4(c) was well attempted but many candidates failed to get the correct answer due to premature rounding errors. However, many other candidates chose the incorrect denominator, or did not find the difference in the numerator showing lack of understanding of the concept of percentage error. In 4d) (i) many candidates achieved the mark for finding the equation of the isocline, but in part (ii) many seemed unsure how to proceed. A small number of strong candidates answered this question very well using a variety of approaches.

## Recommendations and guidance for the teaching of future candidates

It is imperative that students who take the Calculus option have a good foundation in core calculus skills and knowledge.

Teachers are advised to use a range of examples to master the concepts and skills of this option, teach all parts of the syllabus and encourage students to use a variety of problem-solving approaches in the classroom that are time-efficient.

Teachers should continue to emphasize the use correct notation when representing a series. Students need to be dissuaded from making statements such as '$\frac{1}{n}$ is divergent'. Sigma notation is required when reference is made to a series and the series cannot be replaced by its generating sequence.

It is also recommended that teachers:

- Emphasize the conditions for convergence for numerical series and series of functions and provide counter-examples to clarify incorrect applications of the theorems of the course.
- Expose students to questions that require understanding of the fundamental theorem of calculus.
- Explore the concept of isoclines and solve non-routine problems involving these curves.

- Practice more with implicit differentiation involving the use of product rule and learn how to find Taylor series from differential equations
- Encourage attention to details like the modulus is required for the Ratio Test and for the integral of $\frac{1}{x}$ and explain the reasons behind these requirements so that students do not forget it.
- Remind candidates to read questions carefully, answer the questions to the required level of accuracy using appropriate technology and use the "hints" about methods and results provided in the questions.

Finally, teachers are requested to encourage candidates to improve the quality of response to the problems and their writing legibly.

# Higher level paper three sets, relations and groups

## The areas of the programme and examination which appeared difficult for the candidates

On this paper candidates found difficulty with proving group operations on an infinite group, finding equivalence classes, finding inverse functions and working with subgroups.

## The areas of the programme and examination in which candidates appeared well prepared

On the whole candidates appeared to have been reasonably well prepared for questions on composite functions and knowing the basic properties of groups.

## The strengths and weaknesses of the candidates in the treatment of individual questions

### Question 1

Question 1 proved to be a reasonable start to the paper but not an easy start for some candidates. In part a) candidates did not realise that they needed to formally show it was associative and many gave vague written descriptions. In part (c) a number of candidates thought it was enough to show it was Abelian and failed to realise that it was also necessary to consider the other properties of a group. Few recognised that the demonstration of closure was linked to part (b) of the question.

### Question 2

This was attempted by most candidates, but a good understanding of set algebra was needed to make progress. A number of candidates were able to apply De Morgan's law correctly but then failed to explain how the sets combined into the universal set. In part (b) few candidates gained full marks. Many were able to find some elements, but few found them all.

### Question 3

Stronger candidates made good progress with part (a) of the question, but weaker candidates struggled with the formality of what was required. Most candidates found part (b) difficult; they knew to use (1,2) but did not understand the meaning of the result gained.

### Question 4

Candidates started well with this question and part (a) was the most successful question on the paper with many fully correct answers seen. A good number of correct answers were also seen for part (b), but unfortunately the same cannot be said for part (c) where very few candidates were able to find the inverse.

### Question 5

This was found to be a challenging question for this cohort of students. Very few candidates understood what was required in part (a) and very few fully correct answers were seen. There was a little more success in part (b) with candidates understanding the need to show injective and surjective although the formality of what was required was often missing. Very few correct answers were seen for part (c).

## Recommendations and guidance for the teaching of future candidates

- A number of students were let down by not appreciating the level of formality and precision needed in terms of what they write. Within a sets, groups and relations option this is important.
- The quality of presentation on the paper was poor overall. Candidates should remember that in this option especially, they need to explain their thinking to the reader and hence the setting out needs to be clear and logical. This is especially true when asked to “show” results.

# Higher level paper three statistics and probability

## The areas of the programme and examination which appeared difficult for the candidates

It was again seen that some candidates do not appreciate the difference between $\sum_{i=1}^{n} X_i$ and $nX$ regarding the calculation of variance.

Many candidates fail to use their calculators efficiently. This was noted particularly in Questions 2 and 3.

The definition of the probability generating function as $\mathrm{E}(t^X)$ is understood by many candidates, but the algebraic manipulation of this expression sometimes causes difficulty.

## The areas of the programme and examination in which candidates appeared well prepared

Most candidates are confident in solving inferential problems involving hypothesis testing and confidence intervals although in some cases the calculator could be used to a greater extent.

## The strengths and weaknesses of the candidates in the treatment of individual questions

### Question 1

This question was well answered in general. In (b), some candidates wrote incorrectly that $\mathrm{Var}(4X-3Y)=16\mathrm{Var}(X)-9\mathrm{Var}(Y)$ . In (c), some candidates wrote incorrectly that $\mathrm{E}\left(X^2-Y^2\right)=\left[\mathrm{E}(X)\right]^2-\left[\mathrm{E}(Y)\right]^2$ or $\mathrm{E}\left(X^2-Y^2\right)=\mathrm{E}\left[(X-Y)(X+Y)\right]=\mathrm{E}(X-Y)\mathrm{E}(X+Y)$.

### Question 2

In (a), it was surprising that many candidates used the incorrect formula $\frac{\Sigma x^2}{n}-\bar{x}^2$ to find an unbiased estimate for the variance. Candidates who then corrected this by multiplying by $\frac{n}{n-1}$ often used the rounded value of $\bar{x}$ which gave an incorrect answer. The recommended formula to use is $\frac{\Sigma x^2}{n-1}-\frac{(\Sigma x)^2}{n(n-1)}$ which eliminates the possibility of a rounding error. In (b), it was surprising that some candidates calculated the confidence limits manually using the results from (a) rather than using the statistics software on their calculator which was the expected method. Solutions to (c) were sometimes disappointing with some candidates failing to realise that all they had to do was to examine whether 26 was inside each interval.

### Question 3

Almost every candidate solved (a)(i) correctly but the usual mistake was often seen in (a)(ii), this being that the variance was incorrectly calculated as 112 × 82 instead of 11 × 82. In (b), the answer to (i) was often given as the normal distribution, even when the $t$-distribution was used in (ii). Many candidates used a

$z$-test in (ii) which meant using the standard deviation of 8 suggested in (a). This was invalid because the value of 8 was a claim made by Mr Sailor which was not substantiated. The correct approach was to estimate the standard deviation from the data and then use a $t$-test. This, of course, was intended to be carried out on the calculator. In (c), the hypotheses sometimes involved r rather than $\rho$, suggesting that some candidates do not appreciate the difference between the population product moment correlation coefficient and the sample product moment correlation coefficient. Most candidates obtained the correct value of $r$ but, curiously, many candidates then calculated the corresponding value of $t$ and then used the cumulative distribution button on their calculator to find the $p$-value. The mark scheme was written assuming that both $r$ and the $p$-value would be read directly from the calculator. Some candidates failed to give the conclusion in context, as required in the question. In (d), many candidates found the regression line of length on weight instead of weight on length.

### Question 4

In (a), almost all the candidates wrote the correct series for $Gx(t)$, but many went straight from the series to the answer without justification. Since the question was a 'Show that', candidates were required to indicate that the series was geometric and to mention the formula for the sum to infinity. Most candidates solved (b) correctly although in some cases this required many lines of algebra instead of just a few. Part (c) was found to be the most difficult question on the paper. Method 2 on the markscheme, starting with $\mathrm{E}(t^Y)$, was the more popular method for those candidates who attempted this question. Most candidates then wrote this as $t^3\mathrm{E}(t^{2X})$. Candidates who wrote the expectation as $\mathrm{E}[(t^2)^X]$ generally carried on to the correct answer while those who wrote it as $\mathrm{E}[(t^X)^2]$ did not.

## Recommendations and guidance for the teaching of future candidates

Many candidates completely ignore the rubric on the examination paper stating that 'Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures'. Markschemes are sometimes tolerant of deviations from this requirement but candidates should be aware that marks can be lost by giving answers to the wrong level of accuracy.

Candidates should be fully aware of the full capability of their calculator, especially in inferential problems.

Candidates need to be aware that full marks are not necessarily given for correct answers. It is usually essential to give the mathematical justification.