MATHEMATICS HL

Overall grade boundaries

Discrete mathematics

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 24	25 – 36	37 – 49	50 – 63	64 – 76	77 – 100

Series and differential equations

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 25	26 – 36	37 – 49	50 – 63	64 – 76	77 – 100

Sets, relations and groups

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 25	26 – 36	37 – 49	50 – 63	64 – 76	77 – 100

Statistics and probability

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 25	26 – 36	37 – 49	50 – 63	64 – 76	77 – 100

Internal assessment

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 6	7 – 13	14 – 18	19 – 23	24 – 29	30 – 34	35 – 40

Candidates generally produced well-written portfolios, and a few outstanding pieces of mathematical work were noted this session. Generally, the writing was clear and concise. However, there appears to be a tendency on the part of some candidates to produce massive documents that were much too long, using a cut-and-paste approach that was repetitive and tedious.

The assessment criteria appeared to be well understood by both the teachers and the candidates.

The tasks

Virtually all of the portfolio tasks were taken from the current publication, “Mathematics HL – The portfolio – Tasks for use in 2011 and 2012”, with the most popular being “Patterns within Systems of Linear Equations” and “Modelling a Functional Building”. There were very few teacher-designed tasks submitted. Teachers are encouraged to design their own tasks, keeping in mind the need to satisfy all criteria fully.

Candidates’ performance

Most candidates performed well against criterion A. The use of calculator notation was rare. However, some students still insist on providing an unnecessary sequence of calculator keystrokes that are not particularly useful to the reader. Some candidates failed to make a distinction between the terms, “equation”, “function”, and “expression”, or “un” and “u_n”; furthermore, such careless notation was sometimes overlooked by teachers.

Communication skills have improved. However, there were a number of candidates whose work did not flow, particularly when tables and results were relegated to the appendix. Unlabelled graphs were still noted in many pieces of work. Some student work, though correct, was not concise, with some being in excess of 30 pages. Usually, this was the result of a copy-and-paste presentation that often included an excessive number of graphs. Better use should have been made of technology to enhance the presentation of varied examples.

Overall, the candidates produced good work and the assessments against criteria C and D by their teachers have been appropriate. However, in some type I work, the investigation was extremely limited, with no evidence of a pattern to warrant a conjecture, let alone a generalisation, for example, in the “Linear Equations” task.

In type II tasks, care must be taken to ensure that variables are explicitly defined. The analyses of data must be quantified, and if a regression analysis were appropriate, the candidate must have provided reasons for a particular choice. For example, in the “Functional Building” task, many candidates gave no reason for the choice of a quadratic to model the shape of the roof other than that it looked like one! The use of software that automatically determines the “best” regression model leaves little for the candidate to interpret by himself and is of little merit.

Some realisation of the significance of the results obtained in terms of the created model when compared to the actual situation should have been provided, and candidates should have reflected on their findings. A consideration of the scope and limitations of the models developed should have been included, but not at the end of the modelling task as an afterthought.

The use of technology varied considerably. It should be noted that the mere inclusion of graphs is not sufficient to demonstrate a resourceful use of technology. Full marks were often given much too generously for the use of an overwhelming collection of similar graphs. Some graphs were carelessly included that did not support the work, as when a semi-circle was drawn to model a parabolic roof. Should the task require the domain to be limited to positive values, the graphs should have been so adjusted.

On the other hand, the use of sliders to adjust parameters provided an excellent investigative approach. The use of conditional statements in spreadsheets also worked well in the modelling task, “Running with Angie and Buddy”.

There were many good pieces of work. However, the awarding of full marks in criterion F requires more than completion and correctness, but the evidence of mathematical sophistication in a piece of exemplary work. Work that is deemed “very good” that meets but does not exceed expectation should be awarded one mark against criterion F.

Suggestions to teachers

The portfolios in this session could have included tasks taken from the document, “Mathematics HL – The portfolio – Tasks for use in 2011 and 2012”, but not taken from earlier publications. Please note a significant penalty is applied in moderation for the use of “expired” tasks. Teachers are always encouraged to design their own, but to be mindful that such tasks should not be reused year after year.

The portfolios in the sample are expected to contain originals with the teacher’s marks, not unmarked photocopies. Teachers are expected to write directly on their candidates’ work, not only to provide feedback to the candidates, but information to the moderators as well. The use of Form B would allow the teacher to indicate more relevant and descriptive comments.

There has been a noticeable improvement in the provision of background information to each portfolio task that is required to accompany each sample, either on Form A or through anecdotal comments. Moderators find them very useful in determining the context in which the task was given when confirming the achievement levels awarded.

A solution key for tasks from the current publication, as well as for those designed by teachers, must accompany the portfolios in order that moderators can justify the accuracy of the work, and appreciate the level of sophistication demonstrated by the candidates.

For candidates completing the diploma in 2012, the tasks contained in the newest document, “Mathematics HL – The portfolio – Tasks for use in 2012 and 2013”, as well as those in the current document containing “tasks for use in 2011 and 2012”, are eligible for use.

Paper one

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 25	26 – 39	40 – 56	57 – 73	74 – 90	91 – 120

The areas of the programme and examination that appeared difficult for candidates

- Vector notation
- Graph sketching; translating contextual information into mathematics
- Understanding of the logical structure of ‘proof by the method of mathematical induction’, particularly the role of the final summary statement. More generally, there was a lack of appreciation that the most satisfactory solutions require some explanatory commentary.

The areas of the programme and examination in which candidates appeared well prepared

Calculus and probability, including conditional probability.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

The majority of candidates obtained the correct answer. A small minority of candidates used degree measure rather than radian measure, or failed to notice that the triangle was equilateral.

Question 2

A varied response. Many knew how to solve this standard question in the most efficient way. A few candidates expanded $(a+ib)^3$ and solved the resulting fairly simple equations. A disappointing minority of candidates did not know how to start.

Question 3

Candidates who drew a tree diagram, the majority, usually found the correct answer.

Question 4

Parts (a) and (b) were almost invariably correctly answered by candidates. In (c), most errors involved the integration of $\cos(2x)$ and the insertion of the limits.

Question 5

The key to this question was the recognition that the various probabilities were proportional to relevant areas. Most candidates realised this, but made mistakes translating this into correct calculations. Some candidates did not make use of the probability of ½ of not hitting the target.

Question 6

Most candidates were awarded good marks for this question. A disappointing minority thought that the (k+1)th derivative was the kth derivative multiplied by the first derivative. Providing an acceptable final statement remains a perennial issue.

Question 7

Most candidates knew how to tackle this question. The most common error was in giving $+b$ and $-b$ as the x-coordinates of the point of intersection.

Question 8

(a) The fairly easy trigonometry challenged a large number of candidates. Part (b) was very well done. Satisfactory answers were very rarely seen for (c). Very few candidates realised that a minimum can occur at the beginning or end of an interval.

Question 9

(a) The best answered part of the question. The critical points were usually found, but the inequalities were often incorrect. Few candidates were convincing regarding the connection between (a) and (b). This had consequences for (c).

Question 10

Most candidates scored maximum marks on this question. A few candidates found k = -1.

Question 11

This was, disappointingly, a poorly answered question. Some tried to talk their way through the question without introducing the time variable. Even those who did use the distance as a function of time often did not check for a minimum.

Question 12

(a)(i) The majority of candidates were very sloppy in their use of vector notation. Some candidates used Cartesian coordinates, which was acceptable. Part (a)(ii) was well done. Part (b)(i) was usually well started, but not completed satisfactorily. Many candidates understood the geometry involved in this part.

Question 13

(a) Nearly always correctly answered. (b) Most candidates separated the variables and attempted the integrals. Very few candidates made use of the condition $y > 1$, so losing 2 marks. Part (c) was often well answered, sometimes with follow through. Only the best candidates were successful on part (d).

Recommendation and guidance for the teaching of future candidates

- Encourage students to organise their written work in a form that others can understand. A marker can only read what is on the page, but cannot be expected to read the mind of the writer.

- Take care with vector notation and how it is used in abstract situations.
- Work on contextual questions, particularly translating information into mathematical notation.

Further comments

There was a good spread of marks on this paper. Examiners felt that the paper discriminated well throughout the full ability range.

Paper two

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 14	15 – 28	29 – 39	40 – 56	57 – 72	73 – 89	90 – 120

The areas of the programme and examination that appeared difficult for candidates

- Related rates of change
- Conditions for an infinite geometric series to have a finite sum
- Making an accurate sketch from the GDC
- Properties of complex numbers
- Algebraic solution of inequalities
- Gaussian elimination
- Vectors

The areas of the programme and examination in which candidates appeared well prepared

- Binomial, Normal, and Poisson Distributions
- Probability and conditional probability
- Matrices
- Finding the inverse of a function and its domain
- Algebraic manipulation of formulae

The strengths and weaknesses of candidates in the treatment of individual questions

Question 1

Many candidates attempted this question successfully. In (a), however, a large number of candidates did not use the zoom feature of the GDC to draw an accurate sketch of the given function. In (b), some candidates used the domain as the limits of the integral. Other candidates did not take the absolute value of the integral.

Question 2

This question was answered successfully among candidates who chose the determinant method. The algebraic manipulation involved in using row reduction caused difficulties for a number of candidates.

Several candidates did not earn the final ***R*** mark.

Question 3

Parts (a) and (b) were answered successfully by many candidates. Some candidates had difficulty obtaining the correct inequality in (c)

Question 4

Many candidates failed to give the answer for (a) in rational form. The GDC can render the answer in this form as well as the decimal approximation, but this was obviously missed by many candidates. (b) was generally answered successfully.

Question 5

This question was generally answered successfully. Many candidates used the tabular feature of their GDC for (b) thereby avoiding potential errors in the algebraic manipulation of logs and inequalities.

Question 6

Candidates generally found this question challenging. Many candidates had difficulty finding the arguments of $z_1 z_2$ and z_1 / z_2. Among candidates who attempted to solve for z_1 and z_2 in Cartesian form, many had difficulty with the algebraic manipulation involved.

Question 7

A large number of candidates omitted the absolute value sign in the inequality in (a), or the use of the correct double inequality. Among candidates who had the correct statement, those who used their GDC were the most successful. The algebraic solution of the inequality was difficult for some candidates. In (b), quite a number of candidates found the sum of the first *n* terms of the geometric series, rather than the infinite sum of the series.

Question 8

Finding the inverse function was done successfully by a very large number of candidates. The domain, however, was not always correct. Some candidates failed to use the GDC correctly to find (b), while other candidates had unsuccessful attempts at an analytic solution.

Question 9

Few candidates applied the method of implicit differentiation and related rates correctly. Some candidates incorrectly interpreted this question as one of constant linear rates.

Question 10

Many candidates failed to access their GDC early enough to avoid huge algebraic manipulations, often carried out with many errors. Some candidates failed to separate and equate the real and imaginary parts of the expression obtained.

Question 11

Parts (a) and (b) were generally accessible to many candidates. In (c)(i) quite a few candidates missed the wording ‘first throw’, and consequently in (ii) used the incorrect probabilities.

Question 12

Many candidates were able to solve (a) successfully. A few candidates failed to understand the relationship between S_{2n} and S_{3n}, and hence did not obtain the correct equation. (b) was answered poorly by a large number of candidates. There was significant difficulty in forming correct general statements, and a general lack of rigor in providing justification.

Question 13

Candidates generally attempted this question but with varying degrees of success. Although (a) was answered best of all the parts, quite a few did not use correct notation to designate the vector equation of a line, i.e., **r** =, or its equivalent. In (b) some candidates incorrectly assumed the result and worked the question from there. In (c) some candidates did not understand the necessary relationships to make a meaningful attempt.

Question 14

Parts (b) and (c) were answered fairly well by quite a few candidates. In (a) many candidates failed to write the formula for the modulus of a complex number. (c) proved inaccessible for a large number of candidates. The algebraic manipulation required and the recognition of the imaginary and real parts in order to arrive at the necessary relationship were challenging for many candidates.

Recommendation and guidance for the teaching of future candidates

In addition to careful syllabus coverage and extensive practice using past exam questions, teachers are strongly recommended to:

- ensure that students have basic analytical skills and are able to manipulate complicated algebraic expressions
- ensure that students understand how to show or prove results with the appropriate amount of rigor
- ensure that students are competent users of the GDC, and know when to, and when not to, use the GDC in a particular question

- ensure that students have ample opportunities to apply their knowledge and skills both in familiar and unfamiliar settings
- ensure that students show appropriate method when using a GDC, e.g. providing a sketch when answers are obtained from a graph
- ensure that students understand the need for clear and legible working out
- ensure that students understand that any undesired results should be clearly crossed out, and that these will not be considered as part of a solution or answer, even if what was crossed out was correct.

Paper three – Discrete mathematics

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 7	8 – 14	15 – 23	24 – 30	31 – 36	37 – 43	44 – 60

The areas of the programme and examination that appeared difficult for candidates

On this paper candidates found difficulty using Fermat's little theorem.

The areas of the programme and examination in which candidates appeared well prepared

On the whole candidates appeared to have been reasonably well prepared for questions on some aspects of graph theory and using the Euclidean algorithm.

The strengths and weaknesses of candidates in the treatment of individual questions

Question 1

Most candidates were able to draw the graph as required in (a) and most made a meaningful start to applying Prim's algorithm in (b). Candidates were not always clear about the order in which the edges were to be added.

Question 2

Part (a) of this question was the most accessible on the paper and was completed correctly by the majority of candidates. It was pleasing to see that candidates were not put off by the question being set in context and most candidates were able to start part (b). However, a number made errors on the way, quite a number failed to give the general solution and it was only stronger candidates who were able to give a correct solution to part (b) (iii).

Question 3

Most candidates were able to start (a), but many found problems in expressing their ideas clearly in words. Stronger candidates had little problem with (b), but a significant number of weaker candidates had problems working with the concepts of Eulerian circuits and Hamiltonian cycles and with understanding how to find a specific number of walks of a certain length as required in (b) (vii).

Question 4

There were a number of totally correct solutions to this question, but some students were unable to fully justify their results.

Question 5

There were very few fully correct answers. In (b) the majority of candidates assumed that 341 is a prime number and in (c) only a handful of candidates were able to state the converse.

Recommendation and guidance for the teaching of future candidates

- Students need to cover the entire syllabus.
- Students need to know the correct terminology.
- Students need to be aware that both formal and informal proofs can be tested in this option and that a degree of rigour and clarity is required in setting these out.

Paper three – Series and differential equations

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 7	8 – 15	16 – 22	23 – 29	30 – 37	38 – 44	45 – 60

The areas of the programme and examination that appeared difficult for candidates

On this paper candidates found difficulty in finding the interval of convergence, and in correctly using methods of both differentiation and integration.

The areas of the programme and examination in which candidates appeared well prepared

On the whole candidates appeared to have been well prepared for questions on L'Hôpital's rule, finding the radius of convergence of a series and in recognising homogeneous differential equations.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

This question was accessible to the vast majority of candidates, who recognised that L'Hôpital's rule was required. However, some candidates omitted the factor π in the differentiation of $\cot \pi x$. Some candidates replaced $\cot \pi x$ by $\cos \pi x / \sin \pi x$, which is a valid method but the extra algebra involved often led to an incorrect answer. Many fully correct solutions were seen.

Question 2

Part (a) of this question was found challenging by the majority of candidates, a fairly common 'solution' being that the result is true for *n* = 1,2,3 and therefore true for all *n*. Some candidates attempted to use induction which is a valid method but no completely correct solution using this method was seen. Candidates found part (b) more accessible and many correct solutions were seen. The most common problem was candidates using an incorrect comparison test, failing to realise that what was required was a comparison between $\sum \frac{1}{n!}$ and $\sum \frac{1}{2^{n-1}}$.

Question 3

Most candidates were able to start (a) and a majority gained a fully correct answer. A number of candidates were careless with using the absolute value sign and with dealing with the negative signs and in the more extreme cases this led to candidates being penalised. Part (b) caused more difficulties, with many candidates appearing to know what to do, but then not succeeding in doing it or in not understanding the significance of the answer gained.

Question 4

This proved to be a hard question for most candidates. A number of fully correct answers to (a) were seen, but a significant number were unable to integrate $\frac{1}{4x^2+1}$ successfully. Part (b) was found the hardest by candidates with most candidates unable to draw a relevant diagram, without which the proof of the inequality was virtually impossible.

Question 5

Many candidates were successful in (a) with a variety of methods seen. In (b) the use of the chain rule was often omitted when differentiating e^{-y} with respect to *x*. A number of candidates tried to repeatedly differentiate the original expression, which was not what was asked for, although partial credit was given for this. In this case, they often found problems in simplifying the algebra.

Question 6

Most candidates realised that this was a homogeneous differential equation and that the substitution $y = vx$ was the way forward. Many of these candidates reached as far as separating the variables correctly but integrating $\frac{v+1}{v^2+1}$ proved to be too difficult for many candidates – most failed to realise that the expression had to be split into two separate integrals. Some candidates successfully evaluated the arbitrary constant but the combination of logs and the subsequent algebra necessary to obtain the final result proved to be beyond the majority of candidates.

Recommendation and guidance for the teaching of future candidates

- Students need to cover the entire syllabus.
- Students need to have a solid background with skills and understanding in the core calculus portion of the HL programme to be successful with this option.

Paper three – Sets, relations and groups

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 7	8 – 15	16 – 23	24 – 30	31 – 36	37 – 43	44 – 60

The areas of the programme and examination that appeared difficult for candidates

On this paper candidates found difficulty with finding equivalence classes and showing that if *f* and *g* are both injective and surjective, then $f \circ g$ is also injective and surjective.

The areas of the programme and examination in which candidates appeared well prepared

On the whole candidates appeared to have been reasonably well prepared for questions on most aspects of group theory.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

Most candidates were aware of the group axioms and the properties of a group, but they were not always explained clearly. Surprisingly, a number of candidates tried to show the non-isomorphic nature of the two groups by stating that elements of different groups were not in the same position rather than considering general group properties. Many candidates understood the conditions for a group to be cyclic, but again explanations were sometimes incomplete. Overall, a good number of substantially correct solutions to this question were seen.

Question 2

Part (a) was accessible to most candidates, but a number drew incorrect Venn diagrams. In some cases the clarity of the diagram made it difficult to follow what the candidate intended. Candidates found (b) harder, although the majority made a reasonable start to the proof. Once again a number of candidates were let down by poor explanation.

Question 3

Most candidates made a purposeful start to this question and most gained some marks. However, it was only stronger candidates who understood how to demonstrate the general case for closure of the group.

Question 4

Stronger candidates made a reasonable start to (a), and many were able to demonstrate that the relation was reflexive and transitive. However, the majority of candidates struggled to make a meaningful attempt to show the relation was symmetric, with many making unfounded assumptions. Equivalence classes still cause major problems and few fully correct answers were seen to (b).

Question 5

This question was found difficult by a large number of candidates and no fully correct solutions were seen. A number of students made thought-through attempts to show it was surjective, but found more difficulty in showing it was injective. Very few were able to find a single counter example to show that the converses of the earlier results were false. Candidates struggled with the abstract nature of the question.

Recommendation and guidance for the teaching of future candidates

- Students need to cover the entire syllabus.
- Students need to know the correct terminology.
- Students need to understand that they will be penalised for poor explanation or layout of work.
- In this option questions involving proof and demonstration will be asked and it is essential that students understand that a degree of rigour is needed in these proofs and demonstrations.

Paper three – Statistics and probability

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 7	8 – 15	16 – 23	24 – 30	31 – 36	37 – 43	44 – 60

The areas of the programme and examination that appeared difficult for candidates

On this paper candidates found difficulty with working with the cumulative distribution function and the uniform distribution.

The areas of the programme and examination in which candidates appeared well prepared

On the whole, candidates appeared to have been reasonably well prepared for questions on the straightforward use of the normal distribution and the use of the GDC, recognition of the exponential distribution and the generalities of continuous distributions and the basic use of the Chi-squared distribution.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

For most candidates this was a reasonable start to the paper with many candidates gaining close to full marks. The most common error was in (b) where, surprisingly, many candidates did not realise the need to multiply the product of the two probabilities by 2 to gain the final answer. Weaker candidates often found problems in understanding how to correctly find the variance in both (c) and (d).

Question 2

This question also proved accessible to a majority of candidates with many wholly correct or nearly wholly correct answers seen. The most common errors were when candidates thought that this was a Normal distribution with a mean of 0.8 and when candidates attempted to use a t-distribution to solve the problem.

Question 3

For most candidates the question started well, but many did not appear to understand how to find the cumulative distribution function in (b). Many were able to integrate $\lambda e^{-\lambda x}$, but then did not know what to do with the integral. Parts (c), (d) and (e) were relatively well done, but even candidates who successfully found the cumulative distribution function often did not use it. This resulted in a lot of time spent integrating the same function.

Question 4

The majority of candidates were able to start this question and it was pleasing to see that the introduction of a variable in the observed frequency did not put candidates off. Most candidates found an answer to (b) in the correct form, but this was sometimes wrong because candidates had forgotten to combine classes or because they were unable to simplify the algebra correctly. Candidates found (c) more difficult, with a number trying to work with a p-value or the wrong Chi-squared value. The requirement for whole number values at the end of this part was ignored by the majority of candidates who got this far.

Question 5

Candidates found difficulty in this question. Weaker candidates were unable to start and only the very best fully understood the connection between X and U.

Recommendations and guidance for the teaching of future candidates

- Students need to cover the entire syllabus and be prepared for questions on any of the distributions given in the syllabus.
- Students need to have a solid background with skills and understanding in the core statistics and probability section of the HL programme to be successful with this option.