MATHEMATICS HL

Overall grade boundaries

Discrete mathematics

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 13	14 – 26	27 – 39	40 – 50	51 – 63	64 – 74	75 – 100

Series and differential equations

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 25	26 – 37	38 – 50	51 – 62	63 – 74	75 – 100

Sets, relations and groups

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 12	13 – 25	26 – 38	39 – 50	51 – 63	64 – 75	76 – 100

Statistics and probability

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 13	14 – 26	27 – 38	39 – 50	51 – 63	64 – 75	76 – 100

Internal assessment

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 6	7 – 13	14 – 18	19 – 23	24 – 29	30 – 34	35 – 40

Generally, the candidates produced excellent portfolios, and a few outstanding pieces of work were noted this session. The assessment criteria appeared to be well understood by both the teachers and the candidates. Unfortunately, some work contained no teacher marks, and brief, non-descriptive comments provided on the back of Form 5/PFCS were not entirely helpful in the moderation process. Observations made by the moderating team are summarised below.

The tasks:

Most of the portfolio tasks were taken from the current publication*, "Mathematics HL – The portfolio – Tasks for use in 2009 and 2010",* with the most popular being *"Parabola", "Ratios of Areas and Volumes", "Viral Illness",* and *"Freight Elevator".* There were very few teacher-designed tasks submitted. Teachers are encouraged to design their own tasks, keeping in mind the need to satisfy all criteria fully.

Candidates' performance

Most candidates performed well against criterion A. Unfortunately, the use of computer notation such as "^" and "E07" were still in evidence, and some students did not make a distinction between the terms, *"equation"* and *"expression",* or *"un"* and "u_n"*;* furthermore, careless notation was often overlooked by teachers.

Good communication skills were evident in some samples. Such examples contained an introduction to the task of the student's own, and comments, annotations, and conclusions that accompanied the steps and results. Where the work was easy to read and follow, it earned high marks in criterion B. However, there were many candidates whose work did not flow, particularly when there was no introduction to the task or when a question-and-answer format to a task was adopted. Unlabelled graphs and the relegation of tables to the appendix rate poorly in terms of an effective presentation and should have been penalised. Some student work, though correct, was not concise, with some being in excess of 30 pages. Usually, this was the result of copy-and-paste presentation, needlessly accompanied by a graph for every variant to the situation. Better use should have been made of technology to enhance the presentation of varied examples.

Overall, the candidates produced good work, and the assessments against criteria C and D by their teachers were appropriate. However, in some type I tasks, the investigation was cursory, leaving very little reason for a conjecture before the candidate stated a generalisation, for example, in the *"Parabola" task.* In other instances, results were merely quoted from internet sources, for example, in the *"Divisibility"* task, and there was little individual work in exploration and investigation, the key to the type I task.

In type II tasks, care must be given to ensure that variables are explicitly defined. Some realisation of the significance of the results obtained in terms of the created model when compared to the actual situation should have been provided, and candidates should have reflected on their findings. A consideration of the scope and limitations of the models developed should have been included. The analyses of data must be quantified, and if a regression analysis were appropriate, the candidate must have provided reasons for a particular choice. The use of software that automatically determines the "best" regression model leaves little for the candidate to interpret by himself and is of little merit.

The use of technology varied considerably. Full marks were given much too generously for an appropriate but not necessarily a resourceful use of technology, for example, in the inclusion of an overwhelming collection of similar graphs. The use of a slider to adjust parameters would have been much more resourceful. For full marks, the use of technology should contribute significantly to the development of each task, and should not simply be a function of the number of related graphs.

There were many good pieces of work; however, the awarding of full marks in criterion F requires more than completion and correctness, but the evidence of mathematical sophistication in a piece of exemplary work. Work that is deemed “very good” that meets but does not exceed expectation should be awarded one mark against criterion F.

Suggestions to teachers

Tasks from the TSM and earlier publications, and tasks intended for use in “2011 – 2012” must not be used. Please note the significant penalty indicated in the syllabus guide and with the published tasks. Teachers are encouraged to design their own.

The teacher should become fully informed of the portfolio assessment criteria to avoid a significant loss of marks in moderation.

The work in the sample portfolios are expected to be originals with the teacher’s marks, not unmarked copies. Teachers are expected to write directly on their candidates’ work, not only to provide feedback to the candidates, but information to the moderators as well. Some samples contained very few comments, making moderation extremely difficult when it was not possible to determine the basis upon which the teacher awarded marks. The use of Form B would allow the teacher to indicate more relevant and descriptive comments.

The background information to each portfolio task is required to accompany each sample, either on Form A or through anecdotal comments. Moderators find them very useful in determining the context in which the task was given when confirming the achievement levels awarded; however, such information was often missing.

A solution key for tasks from the current publication, as well as for those designed by teachers, must accompany the portfolios in order that moderators can justify the accuracy of the work, and appreciate the level of sophistication demonstrated by the candidates.

The tasks contained in the document, “*Mathematics HL – The portfolio – Tasks for use in 2009 and 2010*”, are now considered to have expired for use in future examination sessions. Candidates completing their diplomas in May 2011 and beyond should not be assigned these tasks.

Paper one

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 14	15 – 28	29 – 42	43 – 56	57 – 70	71 – 84	85 – 120

The areas of the programme and examination that appeared difficult for candidates

- Solving inequalities without using a calculator.
- Manipulation of vectors.
- Mathematical induction.

The areas of the programme and examination in which candidates appeared well prepared

- Binomial expansions.
- Recognition of differential equations of the separable variables type, although the actual integration often caused problems.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

This question turned out to be more difficult than expected. Candidates who squared both sides or drew a graph generally gave better solutions than those who relied on performing algebraic operations on terms involving modulus signs.

Question 2

Most candidates solved this question correctly with algebraic errors being the most common source of wrong answers.

Question 3

Most candidates solved this question correctly with most candidates who explained how they obtained their coefficients using Pascal’s triangle rather than the combination formula.

Question 4

This question was generally well answered with candidates who drew a tree diagram being the most successful.

Question 5

Many candidates had difficulties with this question with the given information often translated into incorrect equations.

Question 6

Many candidates found this question difficult. In (a), few seemed to realise that $u_n = S_n - S_{n-1}$. In (b), few candidates realised that $u_1 = S_1$ and in (c) that S_n could be written as $1 - \left(\frac{a}{7}\right)^n$ from which it follows immediately that the sum to infinity exists when *a* < 7 and is equal to 1.

Question 7

Solutions to this question were often disappointing. In (a), some candidates found the value of *k*, incorrectly, by taking the scalar product of the normal vector to the plane and the direction of the line. Such candidates benefitted partially from follow through in (b) but not fully because their line turned out to be parallel to the plane and did not intersect it.

Question 8

Many candidates separated the variables correctly but were then unable to perform the integrations.

Question 9

Very few correct solutions were seen to (a). Many candidates realised that arccos*x* ≤ $\frac{\pi}{4}$ but then concluded incorrectly, not realising that cos is a decreasing function, that $x \le \cos\left(\frac{\pi}{4}\right)$. In (b) candidates often gave an incorrect domain.

Question 10

To solve this problem, candidates had to know either that (***a*** + ***b***).(***a*** + ***b***) = |***a*** + ***b***|^2 or that the diagonals of a parallelogram whose sides are ***a*** and ***b*** represent the vectors ***a*** + ***b*** and ***a*** – ***b***. It was clear from the scripts that many candidates were unaware of either result and were therefore unable to make any progress in this question.

Question 11

Many candidates knew what had to be done in (a) but algebraic errors were fairly common. Parts (b) and (c) were well answered in general. Part (d), however, proved beyond many candidates who had no idea how to convert the given information into mathematical equations.

Question 12

In (a), only a few candidates gave the correct period but the expressions for velocity and acceleration were correctly obtained by most candidates. In (a)(iii), many candidates manipulated the equation $v = 0$ correctly to give the two possible values for cos(πt) but then failed to find all the possible values of *t*. Solutions to (b) were disappointing in general with few candidates giving a correct solution.

Question 13

Many candidates solved (a) and (b) correctly but in (c), many failed to realise that the equation $x\mathrm{e}^{x} = kx$ has two roots under certain conditions and that the point of the question was to identify those conditions. Most candidates made a reasonable attempt to write down the appropriate integral in (c)(iii) with the modulus signs and limits often omitted but no correct solution has yet been seen to (c)(iv).

Recommendation and guidance for the teaching of future candidates

- Candidates should be encouraged to improve the presentation of their scripts. Some scripts were extremely difficult to follow.
- The correct presentation of proof by induction continues to be poor and candidates need to be given a template for this type of proof.
- Many candidates seemed unaware of certain properties of vectors and this topic needs to be emphasized.

Paper two

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 16	17 – 33	34 – 46	47 – 62	63 – 77	78 – 93	94 – 120

The areas of the programme and examination that appeared difficult for candidates

On this paper candidates found some difficulty with probability, statistics, vector algebra and graphs of functions. Many candidates showed many difficulties in giving answers to the required degree of accuracy, in maintaining accuracy throughout the work to avoid errors due to premature rounding, in providing coherent and concise explanations, in using consistent and appropriate notation and setting their work in a logical manner. A large number of candidates also show difficulties in thinking flexibly and in applying their knowledge to unfamiliar contexts.

The areas of the programme and examination in which candidates appeared well prepared

On the whole, candidates appeared to be well prepared for routine questions and the level of proficiency in using a GDC was also good. Many candidates were also aware of the need of showing work and/or provide reasons to their answers. However there was a significant difference between the performances of candidates in different schools. In some cases, candidates were clearly not prepared to take this examination and did not even attempt to answer many of the questions. On the other hand, it was pleasing to see evidence of excellent teaching in some schools whose candidates knew how to present their work clearly, using appropriate notation and terminology, how to show all the necessary steps in a logical manner and how to take advantage of a GDC to answer questions efficiently and well.

The strengths and weaknesses of candidates in the treatment of individual questions

Question 1

Most candidates attempted this question and part (a) was answered correctly by most candidates but in (b), despite the wording of the question, the obtuse angle was often omitted leading to only one solution.

In many cases early rounding led to inaccuracy in the final answers and many candidates failed to round their answers to two decimal places as required.

Question 2

Most candidates used a GDC to answer this question and many scored full marks in this question. However there were a significant number of candidates who showed little understanding of the meaning of unbiased estimate. In some cases, candidates wasted time by attempting to calculate the required values by hand.

Question 3

Although many candidates were successful in answering this question, a surprising number of candidates did not even attempt it. The main difficulty was in finding the correct z scores. A fairly common error was to misinterpret one of the conditions and obtain one of the equations as $\frac{7-\mu}{\sigma} = -1.155...$. In some cases candidates failed to keep the accuracy throughout the question and obtained inaccurate answers.

Question 4

This implicit differentiation question was well answered by most candidates with many achieving full marks. Some candidates made algebraic errors which prevented them from scoring well in this question.

Other candidates realised that the equation of the curve could be simplified although the simplification was seldom justified.

Question 5

Most candidates attempted this question and used a variety of methods to tackle it. Many were successful in obtaining correct answers. However a few misconceptions related to properties of logarithms were identified (eg $\ln x^3 + \ln y^3 = 5 \Rightarrow x^3 + y^3 = e^5$) and these prevented many candidates from solving the equation correctly.

A small number of candidates used a GDC to find the intersection point of the graphs of the equations and showed a sketch of these graphs.

Question 6

Most candidates attempted this question, using different approaches. The most successful approach was the method of complex conjugates and the product of linear factors. Candidates who used this method were in general successful whereas candidates who attempted direct substitution and separation of real and imaginary parts to obtain four equations in four unknowns were less successful because either they left the work incomplete or made algebraic errors that led to incorrect answers.

Question 7

Most candidates correctly stated the required equation for *m*. However, many algebraic errors in the simplification of this equation led to incorrect answers. Also, many candidates failed to find the value of *m* to the required accuracy, with many candidates giving answers correct to 4 sf instead of 4 dp. In part (b) many candidates did not realize that they needed to calculate $\mathrm{P}(X=1)+\mathrm{P}(X=2)$ and many attempts to calculate other combinations of probabilities were seen.

Question 8

Many candidates answered well this question. Full marks were often achieved. Many other candidates did not attempt it at all.

Question 9

This question was attempted by most candidates who in general were able to find the dot product of the vectors in part (a). However the simplification of the expression caused difficulties which affected the performance in part (b). Many candidates had difficulties in interpreting the meaning of $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ in part (c).

Question 10

Very few candidates answered this question well but among those a variety of nice approaches were seen. This question required some organized thinking and good understanding of the concepts involved and therefore just strong candidates were able to go beyond the first steps. Sadly a few good answers were spoiled due to early rounding.

Question 11

Most candidates with a reasonable understanding of probability managed to answer well parts (a), (b) and some of part (c). However some candidates did not realize that different scores were not equally likely which lead to incorrect answers in several parts. Surprisingly, many candidates completed the table in part c) ii) with values that did not add up to 1. Very few candidates answered part (d) well. The enumeration of possible cases was sometimes attempted but with little success.

Question 12

This was the most successfully answered question in part B, with many candidates achieving full marks. There were a few candidates who misread the question and treated the cube as a unit cube. The most common errors were either algebraic or arithmetic mistakes. A variety of notation forms were seen but in general were used consistently. In a few cases, candidates failed to show all the work or set it properly.

Question 13

This question was well attempted by many candidates. In some cases, candidates who skipped other questions still answered, with some success, parts of this question. Part (a) was in general well done but in (b) candidates found difficulty in justifying that f'(x) was non-zero. Performance in part (c) was mixed: it was pleasing to see good levels of algebraic ability of good candidates who successfully answered this question; weaker candidates found the simplification required difficult. There were very few good answers to part (d) which showed the weaknesses of most candidates in dealing with the concept of asymptotes. In part (e) there were a large number of good attempts, with many candidates evaluating correctly the limits of the integral and a smaller number scoring full marks in this part.

Recommendation and guidance for the teaching of future candidates

Besides covering the entire syllabus and providing extensive practice using past exam questions, teachers are strongly recommended to:

- Ensure that their students have good basic skills and are able to manipulate algebraic expressions easily
- Ensure that students know and understand properties of logarithms and exponentials.
- Ensure that students understand the difference between discrete and continuous random variables.
- Ensure that candidates understand the difference between scalar and vector products and know the properties of these operations.
- Ensure that candidates are aware of the need to work with more significant figures throughout their solutions to problems until their final part of the answer when they should consider what the question requires and give their answer to that degree of accuracy.
- Provide more problem solving practice to ensure that their students are able to apply their knowledge in a wide variety of contexts

- Provide more opportunities for candidates to reflect upon the meaning of concepts to improve their performance in questions that require interpretation due to their abstract nature or unfamiliar context.

Paper three – Discrete mathematics

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 7	8 – 14	15 – 23	24 – 28	29 – 34	35 – 39	40 – 60

The areas of the programme and examination that appeared difficult for candidates

The candidates were less happy when they had to think and to create proofs for themselves e.g. Q.5 than when they were doing known algorithms. It was clear that some (mainly re-sit candidates) were not really prepared for this option at all. This was indicated in Q.1. , where answers such as “this vertex has degree of 90 degrees” and “this other vertex has degree of 60 degrees” were given.

The areas of the programme and examination in which candidates appeared well prepared

The candidates generally knew how to apply algorithms.

The strengths and weaknesses of candidates in the treatment of individual questions

Question 1

(a) This was generally well answered with most candidates knowing that isomorphism did not depend on the label of the vertex.

In (b) most candidates gave the required inequality although some just wrote down both inequalities from their formula booklet. The condition $v \geq 3$ was less well known but could be deduced from the next 2 graphs. Euler’s relation was used well to obtain the quadratic to solve and many candidates could then draw a correct graph.

Question 2

This was a standard Chinese remainder theorem problem that many candidates gained good marks on. Some candidates employed a formula, which was fine if they remembered it correctly (but not all did), although it did not always show good understanding of the problem.

Question 3

Good algorithm work was shown; sometimes there were mistakes in giving the order of the edges chosen by, for example doing Prim's algorithm instead of Kruskal's.

Question 4

Fermat's little theorem was reasonably well known. Not all candidates took the hint to use this in the next part and this part was not done well. Part (c) could and was done even if part (b) was not.

Question 5

Only the top candidates were able to produce logically, well thought-out proofs.

Recommendation and guidance for the teaching of future candidates

Not all candidates were starting a fresh piece of paper for each of the questions.

If a candidate introduces a variable that is not given in the question then they need to say what it stands for so that the examiner can follow their working. Candidates need to be prepared for proofs as well as algorithms and know that vague justifications rarely gain many marks. Looking at the structure of proofs on the mark-schemes of previous exams will help. With many of the points mentioned above, careful corrective marking of a trial exam should have assisted the candidates, if they were prepared to learn.

Paper three – Series and differential equations

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 5	6 – 11	12 – 19	20 – 26	27 – 32	33 – 39	40 – 60

The areas of the programme and examination that appeared difficult for candidates

The most challenging part of the paper was in the selection of the appropriate test to show convergence of a series, and the conditions under which a particular test is valid.

Candidates often failed to use appropriate notation and terminology, e.g., limit notation was often excluded where its usage was important. They often failed to provide full justification for their results.

The areas of the programme and examination in which candidates appeared well prepared

Candidates generally answered successfully questions necessitating L'Hôpital's rule, conditional and absolute convergence, p-series and Maclaurin series.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

Surprisingly, some weaker candidates were more successful in answering this question than stronger candidates. If candidates failed to simplify the expression after the first application of L'Hôpital's rule, they generally were not successful in correctly differentiating the expression a 2^{nd} time, hence could not achieve the final three A marks.

Question 2

This was the least successfully answered question on the paper. Candidates often did not know which convergence test to use; hence very few full successful solutions were seen. The communication of the method used was often quite poor.

a) Many candidates failed to see that this is a telescoping series. If this was recognized then the question was fairly straightforward. Often candidates unsuccessfully attempted to apply the standard convergence tests.

b) Many candidates used the ratio test, but some had difficulty in simplifying the expression. Others recognized that the series is the difference of two geometric series, and although the algebraic work was done correctly, some failed to communicate the conclusion that since the absolute value of the ratios are less than 1, hence the series converges. Some candidates successfully used the comparison test.

c) Although the limit comparison test was attempted by most candidates, it often failed through an inappropriate selection of a series.

Question 3

This was one of the most successfully answered questions. Some candidates however failed to use the data booklet for the expansion of the series, thereby wasting valuable time.

Question 4

Apart from some candidates who thought the differential equation was homogenous, the others were usually able to make a good start, and found it quite straightforward. Some made errors after identifying the correct integrating factor, and so lost accuracy marks.

Question 5

Part (a) was answered well by many candidates who attempted this question. In part (b), those who applied the integral test were mainly successful, but too many failed to supply the justification for its use, and proper conclusions.

Recommendation and guidance for the teaching of future candidates

It was evident that some candidates had weak calculus analytical skills. This option presupposes a strong foundation in core calculus.

The convergence tests need to be better understood in order to be correctly applied. Correct notation and terminology for effective communication needs to be stressed.

Paper three – Sets, relations and groups

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 5	6 – 11	12 – 21	22 – 27	28 – 34	35 – 40	41 – 60

The areas of the programme and examination that appeared difficult for candidates

- Few candidates seemed to know the definition of the Cartesian cross product.
- Candidates generally found it difficult to successfully complete an abstract group theory proof.
- Although definitions and properties were well known and could be accurately stated, candidates found it difficult to apply them successfully in specific examples.

The areas of the programme and examination in which candidates appeared well prepared

Candidates were familiar with the properties of equivalence relations, definitions of surjective and injective functions, and axioms of groups, including cyclic groups.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

Although the properties of an equivalence relation were well known, few candidates provided a counter-example to show that the relation is not transitive. Some candidates interchanged the definitions of the reflexive and symmetric properties.

Question 2

This was the least successfully answered question on the paper. Candidates often could quote the definitions of surjective and injective, but often could not apply the definitions in the examples.

a) Some candidates failed to show convincingly that the function was surjective, and not injective.

b) Some candidates had trouble interpreting the notation used in the question, hence could not answer the question successfully.

c) Many candidates failed to appreciate that the function is discrete, and hence erroneously attempted to differentiate the function to show that it is monotonic increasing, hence injective. Others who provided a graph again showed a continuous rather than discrete function.

Question 3

The candidates who knew the definition of the Cartesian cross product were usually able to answer the question successfully.

Question 4

a) Most candidates had the correct Cayley table and were able to show successfully that the group axioms were satisfied. Some candidates, however, simply stated that an inverse exists for each element without stating the elements and their inverses. Most candidates were able to find a generator and hence show that the group is cyclic.

b) This part was answered less successfully by many candidates. Some failed to find all the elements. Some stated that the order of *ab* is 6 without showing any working.

Question 5

This question was generally answered very poorly, if attempted at all. Candidates failed to realize that the property of closure needed to be properly proved. Others used negative indices when the question specifically states that the indices are positive integers.

Recommendation and guidance for the teaching of future candidates

It is not sufficient that candidates memorize definitions and properties. They need much more practice in using them in specific situations. Although questions determining whether finite sets under a binary operation form a group are generally very accessible, there is a tendency to simply state the general properties rather than showing how they are satisfied for the given set.

Many candidates find proofs with groups difficult, hence this needs more practice.

Paper three – Statistics and probability

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 7	8 – 14	15 – 21	22 – 27	28 – 34	35 – 40	41 – 60

The areas of the programme and examination that appeared difficult for candidates

Candidates were less happy when they had to think and show their understanding of the concepts which occurred in Questions 3, 4 and 5.

The areas of the programme and examination in which candidates appeared well prepared

There was good use of calculators in evidence, with the work being done correctly and applicably.

The strengths and weaknesses of the candidates in the treatment of individual questions

Question 1

Candidates will often be asked to solve these problems that test if they can distinguish between a number of individuals and a number of copies. The wording of the question was designed to make the difference clear. If candidates wrote $w_1 + ... + w_6$ in (a) and 12*w* in (b), they usually went on to gain full marks.

Question 2

The 2 confidence intervals were generally done well by using a calculator. Some marks were dropped by not giving the answers to 2 decimal places as required. Weak candidates did not realise that (b) was a *t* interval. Part (a) (ii) was not as well answered and often it was the first step that was the problem.

Question 3

Realising that this was a problem about the Negative Binomial distribution was the crucial thing to realise in this question. All parts of the syllabus do need to be covered.

Question 4

Poorer candidates just gained the 2 marks for saying what a Type I and Type II error were and could not then apply the definitions to obtain the conditional probabilities required. It was clear from some crossings out that even the 2 definition continue to cause confusion. Good, clear-thinking candidates were able to do the question correctly.

Question 5

Part (a) was generally well answered but there were some surprisingly wrong answers to part (b). Some candidates would clearly have been happier if they could have just used their calculator to obtain the expected values and a *p* value. However this was a question 5 and hence it was testing understanding, although it did say what to do at each stage.

Recommendation and guidance for the teaching of future candidates

Not all candidates were starting a fresh piece of paper for each of the questions.

If a candidate introduces a variable that is not given in the question then they need to say what it stands for so that the examiner can follow their working. It was noticeable that with some schools almost all of their candidates had AP written repeatedly on their scripts but in other schools the majority of candidates escaped without any penalty at all. With many of the points mentioned above careful corrective marking of a trial exam should have assisted the candidates if they were prepared to learn.