MATHEMATICS HIGHER LEVEL

Overall grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0-16	17-31	32-43	44-55	56-65	66-77	78-100

Introduction

This subject report is written by the principal examiners. Each of the authors provides general comments on performance, taking into account the comments of the assistant examiners and team leaders. This report is the only means of communication between the senior examiners and the classroom teachers and should therefore be read by all teachers of Mathematics HL.

The grade award team studied the responses in the G2 forms, the assistant examiners' reports and the grade descriptors (a description of the criteria to be satisfied for each of the individual grade levels) before determining the grade boundaries.

Internal assessment – the portfolio

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0-4	5-6	7-9	10-12	13-14	15-17	18-20

Range and Suitability of Work Submitted

Overall, the range of marks submitted covered a wide spectrum, ranging from 8 to 20 marks. However, the assessment standards also varied considerably.

The majority of the assignments were taken from the Teacher Support Material (TSM) for Mathematics HL, including some from the earlier November 1998 edition. There was also a commendable selection of activities designed by teachers themselves, more so than was the case a year ago.

However, some teacher-designed tasks were lacking in suitable components to satisfy Criterion E. Furthermore, Criterion B was often assessed with little attention being paid to the expected care and detail of presentation.

It appears that some teachers are also submitting activities designed for Mathematical Methods as part of the Mathematics HL portfolio. This may be expedient for teachers who teach a combined Mathematical Methods/Mathematics HL class, but the activities are not at the level of the HL programme and result in a significant loss of marks to the students.

Candidate Performance against each Criterion

Candidates generally performed well against Criterion A (Use of notation and terminology), Criterion C (Mathematical content) and Criterion D (Results or conclusions).

It appears that some candidates were given insufficient guidance in meeting Criterion B (Communication), as noted above.

Criterion E (Making conjectures) was treated inconsistently by teachers and candidates. In several instances, candidates were haphazardly awarded full marks for merely noting a pattern. Many

candidates were not required to engage in formulating a conjecture or presenting a deductive generalization with formal arguments.
Criterion F (Use of technology) varied considerably. The full capabilities of a GDC were generally not realized in the limited scope of the activities given. Full marks were often given for *appropriate* but not *resourceful* use of technology.

Recommendations for the Teaching of future Candidates

- Assignments should be spaced out over the entire time available in the HL programme, that is two years typically. This will help to avoid the problem of having too many assignments based on too narrow a segment of the syllabus over too short a period of time.
- Plan to incorporate the use of the GDC to analyse statistical data, calculate vector solutions, model graphs with matrices, generate pseudo-random data and determine or simulate probabilities, in addition to general graphing purposes in class and for portfolio assignments. Spreadsheet data can now also be processed on a GDC
- Some portfolios contained very poorly presented work. Candidates must be directed to acquire some skill in technical writing. Many candidates have merely shown the steps to the solutions of problems that were severely lacking in explanation, annotation or justification.
- Provide more feedback to candidates on the assignments. Very few assignments contained actual teacher comments for the candidates. Some assignments were entirely devoid of marks or comments. Moderation is extremely difficult when it is not possible to determine the basis on which the achievement levels were awarded by the teacher.

Paper 1

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0-18	19-37	38-53	54-67	68-80	81-94	95-120

No of G2s received: 23

- **Comparison with last year's paper:**

Much easier	A little easier	Similar standard	A little more difficult	Much more difficult
0	2	11	2	1

- **Suitability of question paper:**

	Too easy	Appropriate	Too Difficult
Level of difficulty	0	17	3
	Poor	**Satisfactory**	**Good**
Syllabus coverage	0	15	7
Clarity of wording	0	6	15
Presentation of paper	0	5	14

General Comments

The teacher responses on the G2 form were very favourable. It was felt that the syllabus coverage was good, that opportunity for GDC usage was very good, that the questions were clearly stated and that the paper was of a good standard. There was some comment that the same style of calculator technique was required in several questions and that candidates who are strong in algebra often have no way of showing their skills now with the GDC usage. It was suggested that there should be more of a blend of questions - one or two conceptual questions, but also one or two questions that allow the candidate to demonstrate excellent mathematical skills rather than technological skills.

The assistant examiners were also pleased with the paper but found that candidates had difficulty with Question 2 and Question 20. The examiners found that the examination allowed the candidates to demonstrate a well-rounded knowledge of the syllabus. The candidates demonstrated strengths in algebra, differentiation and progressions. However, there was overwhelming evidence that the areas of the syllabus that continue to present problems for the candidates are probability, trigonometry, rates of change, differential equations and transformations. Examiners felt that the use of the GDC was good and that it seems now that its use has become fairly routine. However, candidates need to be aware that when asked for an exact answer they cannot give an approximate answer obtained by using the GDC. Poor notation and terminology is still common in the work of some candidates. It is inappropriate at this level for candidates to give the equation of the asymptotes as "x is 1 and 4" or "$x \neq 1$ and 4".

Performance on Individual Questions

QUESTION 1 Remainder Theorem

Answers: $a = 4$

Most candidates who used the remainder theorem did so successfully. Those who used long division or synthetic division usually made algebraic errors.

QUESTION 2 Transformations of graphs

Answers: $2x^3 - 9x^2 + 13x - 6$

Solutions were often disappointing with few good answers. Many candidates were quite unaware how to begin. Others defined $g(x) = f(x+1) + 1$.

QUESTION 3 Binomial expansion

Answers: The coefficient of x^3 is -7

Most candidates knew what had to be done. Often parentheses were omitted, the minus sign was forgotten, or the entire term, rather than just the coefficient, was given.

QUESTION 4 Graphs, asymptotes

Answers: $y = 1,\ x = 4,\ x = 1$

Most candidates identified the asymptotes $x = 1$ and $x = 4$ but few gave $y = 1$.

QUESTION 5 Probability

Answers: (a) 0.25

(b) 0.083

Most candidates solved part (a) correctly but then tried to calculate the answer to part (b) as a product of two probabilities.

QUESTION 6 Logarithms

Answers: $1275 \ln 2$

This question was answered well by the majority of candidates. The most common mistake was to assume that the expression was a geometric progression.

QUESTION 7 Composite functions

Answers: (a) $a = -2,\ b = 1$

(b) range is $y \geq 0$

Part (a) was reasonably well answered with only a few candidates combining the function the wrong way round. In part (b), the wrong answer $y > 0$ was often seen.

QUESTION 8 Means, variance

Answers: $a = 5,\ b = 11$

Most candidates were able to show that $a + b = 16$ but few were able to use the variance to obtain a second equation.

QUESTION 9 Solving inequalities

Answers: $-2.30 < x < 0$ or $1 < x < 1.30$

or $-\frac{1}{2}\left(\sqrt{13}+1\right) < x < 0$ or $1 < x < \frac{1}{2}\left(\sqrt{13}-1\right)$

Many candidates ignored the fact that multiplying an inequality does not preserve the inequality unless the factor is positive. This mistake led them to plot $x^3 - 4x + 3$. Candidates who did plot $x^2 - 4 + \frac{3}{x}$ often missed part of the solution because they had an inappropriate window. Those who tried to solve the problem algebraically were usually unsuccessful.

QUESTION 10 Lines of intersection of planes

Answers: $x = t,\ y = t+1,\ z = t$ (or equivalent)

or $\boldsymbol{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ or equivalent

Most candidates knew what to do but algebraic errors were not uncommon. Too many candidates gave an equation of a plane as an answer.

QUESTION 11 Distance, velocity, integration

Answers: 6m

Most candidates simply integrated without realising that the integrand changed sign and therefore obtained 4 m as their answer.

QUESTION 12 Sine and cosine rule, are of triangles

Answers: Area $\Delta\text{ABC} = 2.98(\text{cm}^2)$ (accept 2.97)

Most candidates failed to realise that there were two possible triangles and simply chose the acute value of C and therefore calculated the larger area.

QUESTION 13 Integration by parts

Answers: $\int (\theta\cos\theta - \theta)\mathrm{d}\theta = \theta\sin\theta + \cos\theta - \frac{\theta^2}{2} + c$

Most candidates realised that integration by parts was needed although not all went about it the right way. The constant of integration was often omitted.

QUESTION 14 Points of inflexion

Answers: $x = -2$

Many candidates solved this question correctly. The most common error was thinking that inflexion points occur where the first and not the second derivative is zero.

QUESTION 15 Calculating median

Answers: $m = \sqrt{4-\sqrt{8}}\left(\sqrt{4-2\sqrt{2}}\right)$

Few candidates knew what the median was. Those who did, solved the question well either algebraically or by using their GDC.

QUESTION 16 Volume, differentation

Answers: $\frac{1}{2\pi}(\text{cms}^{-1})$ (do **not** accept 0.159)

Many candidates were unable to do this question, failing to realise that the chain rule had to be used. Some candidates tried to evaluate $\frac{\mathrm{d}r}{\mathrm{d}V}$ instead of the much easier $\frac{\mathrm{d}V}{\mathrm{d}r}$, but were usually unsuccessful.

QUESTION 17 Graphs, minimum values

Answers: $a = 1.33$

Few candidates knew how to begin this question. Many of those who did used the fact that the line they were looking for was orthogonal to the curve.

QUESTION 18 Vectors, dot product

Answers: $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

Few candidates gave a correct solution to this question although some simply guessed the correct answer. Others incorrectly started with $(\boldsymbol{a}+\boldsymbol{b})^2 = (\boldsymbol{a}-\boldsymbol{b})^2$ and expanded this as if $\boldsymbol{a}$ and $\boldsymbol{b}$ were scalars and ended up with the correct answer.

QUESTION 19 Transformations, matrices

Answers: reflection in $y = x\tan\frac{5\pi}{12}$

Many candidates found matrices $\boldsymbol{R}$ and $\boldsymbol{M}$ incorrectly. Those who successfully found them often combined them in the wrong order. Whether they did this correctly or incorrectly, many failed then to recognise the result correctly.

QUESTION 20 Tangents, curves, calculus

Answers: $y = 2\mathrm{e}^x$

Few candidates were able even to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = y$ and very few of those who did realised that they then had to solve a differential equation.

Paper 2

Component Grade Boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0-16	17-32	33-41	42-52	53-63	64-74	75-100

No of G2s received: 24

- **Comparison with last year's paper:**

Much easier	A little easier	Similar standard	A little more difficult	Much more difficult
0	3	11	1	0

- **Suitability of question paper:**

	Too easy	Appropriate	Too Difficult
Level of difficulty	0	20	1
	Poor	Satisfactory	Good
Syllabus coverage	0	13	8
Clarity of wording	0	8	13
Presentation of paper	0	6	15

General Comments

This year there was a sense amongst the senior examiners, which was most definitely supported by the comments received from assistant examiners and teachers, that the balance between the option questions of section B was about right. It will always be impossible to write questions of exactly equal difficulty on the current options but the examiners are paying close attention to this issue.

All examiners noted that overall there was a general improvement in the way candidates followed instructions with regard to the accuracy of their answers.

Examiners are still concerned about the writing of good mathematics in solutions. For example an equation needs a left and right hand side. In this paper too many candidates stated that the equation of a line in vector form is just $\begin{pmatrix}1\\3\\-17\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\5\end{pmatrix}$.

Candidates need to write this as $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\-17\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\5\end{pmatrix}$.

The writing of an integral sign without the dx (or dt or whatever variable is needed) is incorrect notation and, in any case, is ambiguous.

Again it is necessary to remind teachers that candidates need to be taught certain examination techniques. Candidates should always start each question on a fresh page. They should not start answering parts of Question 2 in the middle of the answer to Question 1. At the end of the examination the examination books or pages can then be organized in the examination order, not necessarily the order in which the questions were answered. If a graph is required for a question then it should be attached in that part of the answer book. If another question needs a graph then use an additional piece of paper and attach it to the appropriate section of examination book. Graphs should always be reasonable in size and legible.

Performance on Individual Questions

Section A

QUESTION 1 Sequences, proof by induction

Answers: (a) $u_2 = 4$
$u_3 = 8$
$u_4 = 16$

(b) (i) $u_n = 2^n$

(ii) (a) $\boldsymbol{M}^2 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

$\boldsymbol{M}^3 = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}$

$\boldsymbol{M}^4 = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$

(b) $\boldsymbol{M}^{\boldsymbol{n}} = \begin{pmatrix} n+1 & -n \\ n & 1-n \end{pmatrix}$

This was the question on which nearly all candidates did well. In part (i)(a), most candidates found the first three terms of the given sequence, though some (even good candidates) made minor errors in calculations. In part (b)(i), recognizing the powers of 2 and then forming a conjecture for the n^{th} term was done well. The most common error was conjecturing that $u_n = 2^{n-1}$. Part (b)(ii) was done very well and even candidates who stated wrongly that $u_n = 2^{n-1}$ were able to earn follow through marks.
In part (ii)(a), nearly all candidates used a calculator to find the powers of the matrix ***M***. Most candidates made a correct conjecture in part (ii)(b)(i). It is pleasing to report that candidates have made great improvements in writing a proof by mathematical induction. The only part in which some candidates still appear to be weak is in putting together a complete conclusion that links the various parts of the proof into a final statement.

QUESTION 2 Complex Numbers

Answers: (a) (i) $|z| = 1$

(ii) or $\arg z = \frac{2\pi}{3}$

(c) $\frac{3}{2} + \frac{3\sqrt{2}}{2}\mathbf{i}$

Although this question was not intended to be particularly difficult, candidates certainly found it to be very challenging. Many weaker candidates did not even attempt it. In trying to find the modulus and argument of z in parts (a)(i) and (ii), candidates generally tried to evaluate z in either Cartesian form, or using de Moivre’s theorem. There were many errors made in both calculations. Few candidates realized that the modulus could have been stated immediately and those trying to use the polar form often ignored the minus sign in two of the terms. Some candidates who did find z correctly were unable to state the correct argument. These candidates would have benefited from making a sketch to first illustrate the position of z in the complex plane.

In part (b), showing that z was a cube root of one was impossible for candidates who had an incorrect value of z from part (a). Some candidates found the cube roots of one but did not relate their answers to part (a). Part (c) was poorly done with many calculation errors in the evaluation of the given expression.

QUESTION 3 Graphs, asymptotes, calculus

Answers:

(b) (i) $x = -3$

(ii) $x = 4.39$ $(= e^2 - 3)$

$y = -0.901$ $(= \ln 3 - 2)$

(c) $x = -1.34$ or $x = 3.05$

(d) (ii) Area of $A = \int_0^{3.05} \left(4 - (1-x)^2\right) - (\ln(x+3) - 2)\mathrm{d}x$

(iii) Area of $A = 10.6$

(e) The maximum value is 4.63.

This question was well answered by a very large number of candidates. Candidates in general made very good use of their graphical display calculators. In part (a), the sketches were well done but many candidates failed to show the vertical asymptote at x = -3, even though they stated it later in part (b). Some candidates had a point on x = -3.
In part (b), most candidates used their calculators well to find the intercepts. Some marks were lost by failure to follow the rules on accuracy of answers.
In part (c), there was again good use of the calculator to solve $f(x) = g(x)$.
In part (d)(i), the most common error in sketching the required area was to ignore the $x \geq 0$ condition. In part (d)(ii), there were many correct integrals, the most common error involving how to deal with the region below the x axis. In part (d)(iii), some candidates tried to integrate algebraically, but most used the definite integral capability on their calculator to find the area.
Part (e) was the hardest part of the question with only a few correct solutions. It was very common to see candidates simply assume that the maximum vertical distance occurred at the maximum point of the quadratic.

QUESTION 4 Vectors, parametric form, calculus

Answers:

(i) (a) $x = \lambda + 1, y = -2\lambda + 3, z = 5\lambda - 17$

(b) P(4, -3, -2)

(ii) (b) $\ln x + \ln 5 = \ln\left(\frac{y^2}{x^2} + 1\right)$ or $y = x\sqrt{5x-1}$

This question covered two separate topics. The vector geometry was done well but solving a differential equation seems to be an area of the programme that is not well understood.
In part (i)(a), most candidates found the equation of the line but not necessarily in the required parametric form. In part (i)(b), most candidates did an excellent job using the zero scalar product as the necessary condition for point P.

In part (ii)(a),(b) many candidates used the given substitution to obtain the differential equation in terms of v. However the left hand side was not always dealt with as well as the right hand side. Many candidates were unable to separate the variables to solve the equation. It was not uncommon to see candidates actually substituting numbers straight into the differential equation leading to meaningless numerical values.

QUESTION 5 Probability

Answers: (a) $\frac{6}{17}$ $(= 0.353)$

(b) $\frac{5}{21}$ $(= 0.238)$

(c) 0.442 (Accept 0.443)

Question 5, along with Question 2, turned out to be the hardest questions on this examination paper. Probability is still an area of the program that appears to give candidates many problems.

In part (a)(i), many candidates successfully showed that A and B were not independent but then continued in part (ii) as if the events were independent.

In part (b), many candidates considered correctly the various combinations of two electricians and one plumber but forgot that two others had to be chosen from the remaining five.

There were many candidates who made no attempt at part (c), although candidates familiar with the normal distribution often gave completely correct solutions.

Section B

QUESTION 6 Statistics

Answers: (i) 0.1847 (accept 0.1848)

(ii) [2.703, 2.707]

(iii) (a) (i) At the 5% level, we must accept H_1

(ii) At the 1% level, there is not enough evidence to conclude that the die is not fair (and hence we accept H_0)

(iv) accept H_0

Part (i) involving the Poisson distribution was successfully completed by most candidates.

In part (ii), many candidates did not realize that the t-distribution was required and instead used the normal distribution..

The chi-squared test in part (iii) was generally well done but some candidates did not state the hypotheses and explanations of "level of significance" were often very weak.

Part (iv) was done very badly with only a few candidates recognizing the contingency table required.

QUESTION 7 Sets, Relations and Groups

Answers: (i) (a)

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(b)

*	*a*	*b*	*c*	*d*
a	*b*	*a*	*d*	*c*
b	*a*	*b*	*c*	*d*
c	*d*	*c*	*a*	*b*
d	*c*	*d*	*b*	*a*

(ii) (b) (ii) $\{5,10\}$
$\{1,4,6,9\}$
$\{2,3,7,8\}$

(iv) (a) $\begin{pmatrix} a & b & c & d \\ b & d & a & c \end{pmatrix}$

(b) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}; \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$

(c) $\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$
$\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}; \begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}; \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$

This was the most popular Section B question.
Parts (i)(a) and (i)(b) were done very well.
In part (ii)(a), there were many excellent solutions proving that R was an equivalence relation.
In part (b), the definitions of what is meant by "the equivalence class containing a" were often very poor and few candidates were able to list the equivalence classes.
There were very few attempts at part (iii). Confusion between subsets and subgroups seemed to be a problem with some candidates even quoting Lagrange's Theorem.
Many candidates scored well in part (iv) although attempts to show in part (c) that the elements form a subgroup was often incomplete.

QUESTION 8 Discrete Mathematics

Answers:

(i) (b) $d = 14$
$x = 3$ and $y = -7$

(ii) $y_n = \frac{\sqrt{3}}{12}\left(\left(1+\sqrt{3}\right)^{n+1} - \left(1-\sqrt{3}\right)^{n+1}\right)$
(or $y_n = 0.144(2.73^{n+1} - (-0.732)^{n+1})$)

(iii) (b) yes

(c) yes

(v) (b) $\mathrm{P} \to \mathrm{T} \to \mathrm{W} \to \mathrm{X} \to \mathrm{Y} \to \mathrm{U} \to \mathrm{R} \left\{ \begin{matrix} \mathrm{S} \\ \mathrm{Q} \end{matrix} \right.$

There were very few attempts at this question.
In part (i), explaining and using Euclid's algorithm for finding the greatest common divisor was well done by many candidates.
Candidates found the algebra required to solve the recurrence relation in part (ii) to be a challenge.
In part (iii), candidates showed good understanding of graph theory.
Proofs offered in part (iv) often lacked sufficient justification.
In part (v), descriptions of the process were often weak but there were many completely correct solutions using the depth-first algorithm for finding a spanning tree.

QUESTION 9 Analysis, approximation

Answers:

(i) (b) (ii) $\pi \approx 3.14159$

(ii) (c) (i) $k = 1.318311$

(iii) $3.1415926146 < \pi < 3.141637847$

(iv) $\pi = 3.142$

This was the second most popular option question. However candidates sometimes struggled through the various parts, not following the way the question was trying to lead them.
Many candidates made part (i)(a) much harder than necessary. Candidates need to be made aware that the first part of a question with only a 1 mark allocation is meant to be quick and easy.
In part (b)(i), no candidate was able to give a complete justification as to why the method converged. In part (ii) there were many reasonable attempts to apply the Newton-Raphson method.
In part (ii)(a), showing that the fourth derivative was bounded was done quite well.
Most candidates found part (ii)(b) straightforward.
In part (c)(i), although the method was known, many candidates had great difficulty calculating a correct approximation to A.
In part (c)(ii), only a few candidates were able to refer to the error term for Simpson's rule.

Few candidates had any idea how to proceed in parts (iii) and (iv).
In part (iii), the work on the sequence of partial sums was not at all well done. In part (b) some candidates scored marks with alternative proofs with regard to the divergence of the harmonic series.

QUESTION 10 Euclidean Geometry, conic sections

Answers: (i) (a) parabola

(b) $m^2 - 18m - 16c + 161 = 0$

(iii) (a) $n = \dfrac{(a^2 - b^2)\cos\theta}{a}$

(b) $\text{PF}_1 = |a + c\cos\theta| = a + c\cos\theta$
$\text{PF}_2 = |a - c\cos\theta| = a - c\cos\theta$

(c) $\text{NF}_1 = \left(\dfrac{c}{a}\right)(a + c\cos\theta)$
$\text{NF}_2 = \left(\dfrac{c}{a}\right)(a - c\cos\theta)$

Only a very small number of candidates attempted this question. It was either done very well or very poorly.
Most candidates recognized the conic in part (i)(a) as a parabola. In part (i)(b), only a few good candidates were able to find the equation of the tangent and then the relationship between *m* and *c*.
In part (ii), amongst the good candidates, there were some excellent solutions using Ceva's Theorem.
Some good answers were seen to part (iii). Some candidates found the equations in parts (b) and (c) rather than the required distances.

